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MÉMOIRE D'HABILITATION À DIRIGER DES RECHERCHES

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Singularités, feuilletages et applications à la géométrie quasi-analytique et sous-riemannienne

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Divers :

- J'ai coordonné le projet [BdSE19], en collaboration avec José Ginés Espín Buendía, qui a été inclus comme l'un des résultats de sa thèse de doctorat [Es17].

Table des matières

Table des matières **v**

1 Synthèse et objectifs **2**
1.1 Contexte global 2
1.2 Résolution des singularités et log-différentielles 6
1.3 Géométrie réelle quasi-analytique 8
1.4 Géométrie sous-Riemannienne et la Conjecture de Sard . . . 10
1.5 Perspectives 12

2 Listes des travaux **16**
2.1 Listes des travaux présentés pour l'habilitation 16
2.2 Liste complète des travaux par thèmes 17

3 Preliminaries **20**
3.1 Analytic spaces and ideal sheaves 20
3.2 Resolution of singularities 22
3.3 Local resolution of singularities 26
3.4 Singular foliations 27
3.5 Reduction of singularities of planar line foliations 28
3.6 Quasianalytic classes 30
3.7 Properties of quasianalytic classes 34

4 Resolution of singularities and log-differentials **38**
4.1 Introduction 38
4.2 Resolution of singularities of the cotangent sheaf 39
4.3 Monomialization of Darboux first integrals 42
4.4 Monomialization of quasianalytic morphisms 43
4.5 Logarithmic Fitting ideals 48
4.6 Resolution of singularities adapted to log-differentials 50

5 Real quasianalytic geometry **60**

5.1 Introduction 60
5.2 Solution of quasianalytic equations 61
5.3 Composition of quasianalytic maps 63
5.4 Sharp estimates for blowing down 63
5.5 Applications of the monomialization of quasianalytic morphisms 64
5.6 Quasianalytic continuation 67

6 Sard Conjecture in SR-geometry 74
6.1 The Sard Conjecture 74
6.2 Sard Conjecture in dimension 3 77
6.3 The Sard Conjecture over Martinet surfaces 79
6.4 Interlude: Semi-analytic sets 81
6.5 Strong Sard Conjecture in dimension 3 82
6.6 Characteristic line foliation 83
6.7 Divergence of the characteristic vector-field 83
6.8 Sketch of proof when the Martinet surface is non-singular . . 85
6.9 Resolution of singularities of the characteristic foliation . . . 87
6.10 Singular horizontal paths in dimension 3 89

Bibliographie 94

Chapter 1

Synthèse et objectifs

1.1 Contexte global

O sapo não pula por boniteza,
mas porém por precisão.

Guimarães Rosa

Nous allons présenter dans ce mémoire une partie de nos résultats obtenus depuis Septembre 2013, principalement en collaboration avec Edward Bierstone et Ludovic Rifford; la liste des collaborateurs inclut aussi Iwo Biborski, Michael Chow, Alessio Figalli, Vincent Grandjean, Avner Kiro, Pierre Milman et Adam Parusiński. Ces travaux touchent à des questions très diverses dans le vaste domaine des *équations différentielles et singularités* en géométrie algébrique et géométrie différentielle, mais ont tous en commun leur utilisation de la théorie de résolution des singularités combinée avec des méthodes analytiques ou différentielles.

En géométrie, on s'intéresse à des problèmes *globaux* et des problèmes *locaux* mais, en pratique, il est souvent nécessaire de résoudre simultanément plusieurs problèmes locaux et globaux de manière interconnectée. En substance, une *singularité* désigne tout point d'un object géométrique, comme d'une variété, d'un morphisme, d'une forme différentielle ou d'un feuilletage, en lequel il y a un comportement local non-trivial (et qui, comme l'a remarqué Hawking pour les trous noirs, peut cacher en soi des problèmes globaux). En analyse, géométrie, physique, ou encore d'autres sciences, les singularités sont inévitables. Elles apparaissent, par exemple, chaque fois qu'il y a changement brusque d'état ou de comportement en rapport avec un certain paramètre. Pour des valeurs génériques, en général, les aspects qualitatifs (par exemple topologiques, différentielles, etc) restent stables par petits changements de la valeur des paramètres. Cependant, pour des valeurs exceptionnelles, les aspects qualitatifs du phénomène peuvent changer soudainement lorsqu'on opère une faible variation. Ce changement brusque est appelé bifurcation, catastrophe, métamorphose, etc, dans différentes branches des sciences. Des exemples typiques sont la transformation d'eau en glace, le battement du cœur (hysteresis), les caustiques en optique, etc.

Les points singuliers des équations différentielles, en particulier, sont aussi intrigants que fréquents. Depuis l'introduction du calcul différentiel au XVIIème siècle, les *équations différentielles* en général, et les *feuilletages* en particulier, sont étudiés de manière intensive car ils jouent un rôle crucial en mathématiques, en ingénierie, en informatique et en sciences naturelles, y compris la physique et la biologie. Pourtant, même les plus simples feuilletages, comme celui généré par un champ de vecteurs polynomial sur $\mathbb{R}^2$, cachent des singularités très complexes. C'est étonnant, par exemple, que le 16ème problème de Hilbert reste ouvert même lorsque les polynômes sont quadratiques ! Cette complexité se reflète dans la richesse des différentes méthodes utilisées pour les étudier : formes normales, stabilité structurelle, stratifications, résolution des singularités, transformées de Borel-Laplace (et leurs variations), expansion asymptotique, quasi-analyticité, etc. Parmi ces techniques, nombre d'entre elles ont leur origine dans l'étude de points singuliers d'une autre nature, tels que ceux apparaissant en géométrie algébrique.

La *résolution des singularités* est l'une des méthodes les plus efficaces et accomplies pour étudier les singularités en géométrie algébrique. Une liste impressionnante d'applications figurait déjà dans le discours de Grothendieck à l'ICM de Nice en 1970, où Hironaka a reçu la médaille Fields. Philosophiquement, l'idée est de transformer un *problème local très difficile* en un *problème global qui est localement simple.* Considérons, par exemple, une hypersurface $X = \{x \in \mathbb{C}^n;\, P(x) = 0\}$, où P est un polynôme; l'ensemble singulier $\mathrm{Sing}(X)$ de X est constitué de tous les points où X n'est pas une variété différentielle. Une résolution des singularités de X produit un morphisme propre birationnel $\sigma : \widetilde{X} \to X$ où la variété $\widetilde{X}$ est partout lisse. Mais cela est en général insuffisant pour comprendre le comportement du point singulier (par exemple la topologie locale de X) car on ne contrôle pas la structure globale de la pré-image de l'ensemble singulier. Nous avons besoin de plus qu'une simple “disparition des singularités” ou “paramétrisation”. Plus précisément, le morphisme σ est une bonne résolution des singularités si l'image inverse des singularités $E := \sigma^{-1}(\mathrm{Sing}(X))$ est un diviseur à *croisement normaux simples.* En particulier, la résolution capture des *informations locales* et les code dans une *structure combinatoire globale.*

Pourtant, pendant près de quarante ans, la technique n'a été accessible qu'à un petit groupe de spécialistes, ce qui a considérablement limité sa portée. Depuis le début des années 2010 la situation a radicalement changé. S'appuyant sur l'effort de Bierstone, Cutkosky, Encinas, Kollár, Milman, Mustaţă, Villamayor, Włodarczyk, entre autres, la technique est désormais totalement transparente et accessible même aux étudiants de master. Il est désormais possible de modeler la technique, de la plier, de la mélanger avec des idées et des méthodes de différentes saveurs et domaines. Cette flexibilité est exploitée à travers chacun des trois thèmes de ce mémoire :

(I) Résolution des singularités et log-dérivées, voir §1.2;
(II) Géométrie réelle quasi-analytique, voir §1.3;
(III) Géométrie sous-Riemannienne et la Conjecture de Sard, voir §1.4;

Dans chaque sujet, soit les objectifs, soit les méthodes, sont liés aux équations différentielles et à l'analyse classique. Le premier thème concerne le développement des techniques de résolution des singularités, notamment la *résolution des singularités adaptées aux log-dérivées*, dans le contexte des formes différentielles et feuilletages, et leurs applications dans l'étude du *faisceau cotangent*, des *intégrales premières* et des *morphismes*. Le deuxième thème concerne l'étude des propriétés algébriques et géométriques des classes quasi-analytiques. La notion de quasi-analyticité a été partiellement motivée par des équations différentielles, et nous utilisons de nouvelles méthodes analytiques et différentielles, notamment le *prolongement quasi-analytique* et la *monomialisation des morphismes quasi-analytiques*, pour étudier des classes C^∞ quasi-analytiques. Enfin, le troisième thème concerne l'étude de la conjecture de Sard en géométrie sous-riemannienne, qui est considérée comme l'un des deux principaux problèmes ouverts dans le domaine, l'autre étant l'étude de la régularité des géodésiques sous-riemanniennes. Dans nos travaux, nous avons introduit en géométrie sous-riemannienne des techniques de résolution des singularités et d'équations différentielles, en suivant l'approche développée pour le problème de Dulac. Cela nous a permis de fournir une preuve de la version forte de la conjecture de Sard pour les variétés analytiques tridimensionnelles. En combinant notre résultat avec d'autres travaux récents, nous avons montré que, sous la même hypothèse, toutes les géodésiques sous-riemanniennes sont C^1. Je fournis des détails sur l'évolution de ces projets dans §1.5.

Exposons brièvement ce que ce mémoire n'inclut pas, en particulier les travaux récents [BdSCR20], [BdSE19], [BdSG20], [BdSFP19], [BdSFP20]. Au cours des deux dernières années, mes intérêts de recherche se sont élargis vers de nouvelles directions qui vont au-delà du thème général qui conduit ce mémoire. Dans notre travail conjoint avec Curmi et Rond [BdSCR20], nous fournissons une preuve complète du Théorème des rangs de Gabrielov, un résultat fondamental en géométrie analytique locale dont la preuve originale est considérée comme très difficile. Dans notre travail conjoint avec Espín Buendía [BdSE19], nous fournissons une classification topologique de tous les ensembles périodiques limites des familles de champs de vecteurs planaires analytiques. Ce travail fait partie de la thèse de doctorat de Espín Buendía. Dans notre travail conjoint avec Gazeau [BdSG20], nous introduisons quelques techniques d'équations différentielles en apprentissage automatique (en anglais,“Machine Learning”), afin d'étudier l'algorithme d'optimisation ADAM qui a été introduit en 2014. ADAM a eu un grand succès empirique en “deep learning” et est largement utilisé dans l'industrie, mais reste un mystère d'un point de vue théorique. Enfin, dans nos travaux conjoints avec Fantini et Pichon [BdSFP19], [BdSFP20], nous combinons le formalisme provenant des entrelacs non-archimedians avec les projections génériques, la résolution des singularités et la topologie afin d'étudier la métrique interne et les configurations polaires d'une surface singulière.

Un mot sur le style que nous avons adopté pour la rédaction de ce mémoire. L'objectif principal de ce texte est de fournir au lecteur un panorama des idées clés de mes résultats, étayé par des références précises. L'objectif n'est pas de donner une présentation technique complète. Pour certains résultats clés, je vais esquisser les preuves sous des hypothèses supplémentaires afin de mettre en évidence les idées principales, tout en gardant la présentation élémentaire. Dans la mesure du possible, je suis la philosophie que j'ai apprise avec Ludovic Rifford : “les techniques doivent être au service des idées, et non l'inverse”. Enfin, la présentation se veut accessible aux étudiants de master et aux doctorants.

C'est enfin un grand plaisir de remercier les nombreuses personnes qui ont soutenu mes projets scientifiques tout au long de ces années, à commencer par mes plus proches collaborateurs Edward Bierstone et Ludovic Rifford. Je suis également très reconnaissant à tous mes collègues d'Aix-Marseille, spécialement à Anne Pichon, Guillaume Rond, Erwan Rousseau et David Trotman, pour m'avoir accueilli chaleureusement et pour l'agréable ambiance de travail. Je remercie également Felipe Cano, Dominique Cerveau, Georges Comte, Vincent Grandjean, Alessio Figalli, Frank Loray, Pavao Mardesic, Jean-François Mattei, Daniel Panazzolo, Adam Parusińksi, Julio Rebelo, Ana Reguera, Jean-Phillippe Rolin et Tamara Servi pour leur soutien au cours des sept dernières années, surtout au début de ma carrière. Certains d'entre eux ne savent peut-être même pas que certaines courtes interactions ont eu un impact important. Je tiens à remercier aussi mes amis et collaborateurs Octave Curmi, José Espin Buendia, Lorenzo Fantini, Maxime Gazeau et Avner Kiro, qui ont contribué à faire du travail

une activité très heureuse. Enfin, je m'excuse d'avance auprès de tous ceux que j'ai oublié de remercier en cette chaude après-midi d'août. Soyez assuré que, même si la mémoire échoue, le cœur est tout de même reconnaissant.

1.2 Résolution des singularités et log-différentielles

La résolution classique des singularités d'une variété algébrique X (sur un corps de caractéristique zéro) produit une suite finie

$$X = X_0 \xleftarrow{\sigma_0} X_1 \xleftarrow{\sigma_1} \cdots \xleftarrow{\sigma_r} X_r = \widetilde{X},$$

où la variété $\widetilde{X}$ est partout lisse et chaque morphisme σ_i est un *éclatement lisse* (satisfaisant une condition de compatibilité supplémentaire avec les morphismes précédents de la suite). Il s'agit d'une opération birationnelle très simple qui correspond à la paramétrisation par les coordonnées polaires en dimension deux. La résolution des singularités est une méthode puissante pour capturer des informations sur les singularités des variétés et les coder dans une structure combinatoire globale, un diviseur E à *croisement normaux simples* (SNC).

Toute l'histoire de la géométrie algébrique est intimement liée au développement de la résolution des singularités. L'un des premiers résultats profonds de la géométrie algébrique est la preuve de Newton et Puiseux d'une "résolution des singularités" pour les courbes dans le plan complexe. Au cours des deux siècles suivants, plusieurs travaux ont été consacrés à la résolution des courbes et des surfaces. Après beaucoup d'efforts, Zariski a pu prouver une version *locale* pour des variétés arbitraires via des techniques de valuation dans [Z40], et une version globale pour X de dimension trois dans [Z44], par recollements à partir de son résultat local. Le cas général (sur un corps de caractéristique zéro) a été prouvé par Hironaka dans [Hi64], via une approche complètement nouvelle, basée sur les *faisceaux d'idéaux* $\mathcal{I}$ d'une variété ambiante lisse M, dont $Z(\mathcal{I})$ (l'ensemble où toutes les fonctions dans $\mathcal{I}$ sont zéro) est égal à X.

Depuis au moins le début du XXe siècle, les mathématiciens s'intéressent à étendre la résolution des singularités vers d'autres contextes liés aux équations différentielles. Par exemple, la résolution des singularités des feuilletages est l'ingrédient fondamental pour plusieurs problèmes classiques et modernes dans la théorie de systèmes dynamiques et la géométrie algébrique, par exemple le 16ème problème de Hilbert [Du1923], [DR91] [Rou95] et le Programme de Modèle Minimal pour les feuilletages [McQ08]. Une motivation plus subtile apparaît lorsque les équations différentielles ont un rôle *indirect.* Par exemple, Giraud soutient la thèse qu'une preuve de résolution des singularités en caractéristique positive exigerait un contrôle considérable des formes différentielles [Gi83]; et il y a une accumulation d'indices que le contrôle des *dérivées tangentes a un morphisme* est crucial lorsqu'il s'agit de monomialisation du morphisme et de la résolution des singularités en famille [BdSB19] [ATW20a] [ATW20b].

Malheureusement, les équations différentielles ne se transforment pas bien par éclatements, et les résultats dans la plupart des travaux mentionnés ci-dessus sont soit *locaux*, soit limités aux *petites dimensions*, jusqu'à trois. En effet, considérons un sous-faisceau (cohérent) Δ de dérivations, ou champs de vecteurs. Contrairement aux faisceaux d'idéaux, le tiré en arrière des éléments de Δ par un éclatement est *asymétrique.* Par exemple, considérons $\mathbb{C}^3$, le système de coordonnées (x_1, x_2, x_3), et le sous-faisceau de dérivations Δ engendré par les deux champs de vecteurs réguliers $(\partial_{x_2}, \partial_{x_3})$. Soit $\sigma : M \to \mathbb{C}^3$ l'éclatement de centre $(x_1 = x_2 = 0)$, et considérons la carte x_1 (qui est donnée par $x_1 = y_1$, $x_2 = y_1 \cdot y_2$ et $x_3 = y_3$) où le pull-back de Δ est engendré par:

$$\sigma^*(\Delta) = \left(\frac{1}{y_1} \partial_{y_2}, \partial_{y_3} \right).$$

Remarquons qu'il y a une asymétrie entre les pull-back des générateurs : un pôle apparait dans le pull-back de ∂_{x_2}, mais pas dans le pull-back de ∂_{x_3}.

En raison de cette asymétrie, il semble difficile d'adapter aux dérivations, de manière directe, l'approche utilisée par Hironaka. Le manque de compréhension de ce phénomène d'asymétrie est à l'origine de certains faux pas. De plus, les progrès récents sur la réduction des feuilletages reposent sur des éclatements pondérés [Pan06, McQP13] (qui permettent de "compenser" l'asymétrie) ou sur des techniques de valuation [CRS15, CD18] (qui contrôlent l'asymétrie par des méthodes combinatoires), et la situation est similaire, par exemple, pour la réduction des morphismes [Cu99, Cu02, Cu05, Cu07, Cu15, Cu17, BdSB19, ATW20a, ATW20b].

Au cours des sept dernières années, j'ai travaillé avec des techniques de *résolution de singularités adaptées aux log-différentielles*, c'est-à-dire, des méthodes qui prennent en considération le comportement asymétrique d'un certain ensemble de formes différentielles logarithmiques après éclatement, voir §4.6. Le développement de ces techniques a été fortement influencé par les travaux de Bierstone et Milman [BM08], Cutkosky [Cu02] et Denkowska et Roussarie [DR91]. J'ai utilisé des variantes de ces techniques pour démontrer des résultats sur : la résolution du faisceau cotangent en collaboration avec Bierstone, Grandjean et Milman [BdSBGM17] [BdSB17], voir §4.2 ; la réduction des intégrales premières [BdS18], voir §4.3 ; et la monomialisation des morphismes quasi-analytiques en collaboration avec Bierstone [BdSB19], voir §4.4. J'ai également appliqué certaines de ces techniques dans mes travaux sur la géométrie sous-Riemannienne [BdSR18] [BdSFPR18], voir §1.2 et §6.9, où l'on doit utiliser plusieurs résultats de résolution de singularités (de variétés, métriques et feuilletages) d'une façon compatible.

Les préliminaires sur la résolution des singularités se trouvent dans le chapitre 3. Dans le chapitre 4 je donne un aperçu de nos résultats, et je présente les idées principales derrière les techniques des résolution de singularités adaptées aux log-différentielles.

Remarque : Récemment, je suis rentré en contact avec les nouveaux travaux d'Abramovich, Temkin et Włodarczyk [ATW20a] [ATW20b] sur la désingularisation des idéaux relative aux morphismes logarithmiquement réguliers, et leur relations avec la monomialization des morphismes. Suite à une communication personnelle avec les auteurs, je comprends qu'il serait très intéressant de faire une comparaison de leur technique avec les méthodes que nous avons utilisées dans [BdSB19].

1.3 Géométrie réelle quasi-analytique

La notion de quasi-analyticité remonte, au moins, au début du XXème siècle [Denj1921] [Car1926], c.f. [Hol1901], et s'est développée en étroite relation avec les équations différentielles et, plus récemment, avec la géométrie modérée (i.e. tame). En bref, un ensemble de fonctions $\mathcal{C}$ est quasi-analytique si on peut lui associer une notion de développement asymptotique (par exemple, le développement de Taylor) qui est injective. La liste des sujets où la quasi-analyticité intervient est très vaste. Pour n'en citer que quelques-uns, elle inclut l'étude des EDP linéaires, à la suite des travaux de E. Borel, E. Holmgren et Hadamard,[Bo1900], [Hol1901], [Ha1923], l'étude générale des transitions de Poincaré avec des applications au problème de Dulac et au 16ème problème de Hilbert, à la suite des travaux d'Ilyashenko et Ecalle [I91] [Ec92] (voir aussi [Sp18]), et l'étude des structures o-minimales, à la suite des travaux de Rolin, Speissegger et Wilkie [RSW03] (voir aussi [Mi95] et [RS15]).

Au cours des cinq dernières années, j'ai travaillé avec les *classes quasi-analytiques*, une notion générale de fonctions C^∞ quasi-analytiques fermées relativement à certaines propriétés algébriques. Plus précisément, les classes quasi-analytiques réelles sont des classes de fonctions réelles infiniment différentiables (c'est-à-dire que pour chaque ensemble ouvert $U \subset \mathbb{R}^n$ nous associons une sous-algèbre $\mathcal{Q}(U) \subset C^\infty(U))$ caractérisées par trois axiomes: la *quasianalycité*, c'est-à-dire, l'injectivité de l'homomorphisme donné par la série de Taylor, en tout point ; la fermeture par le théorème de fonction implicite ; et la fermeture par division d'une coordonnée ; voir §3.6. La définition axiomatique des classes de fonctions quasi-analytiques est très récente [BM04, RSW03]. En particulier, elle capture les conditions minimales pour qu'une classe de fonctions C^∞ admette une *résolution des singularités* [BM97, BM04]. Elle englobe plusieurs classes classiques de fonctions qui ont été étudiées en analyse réelle, en théorie des équations différentielles partielles et en théorie des modèles. Il y a deux exemples généraux de classes quasi-analytiques qui présentent un intérêt particulier pour ce mémoire :

(I) **Les classes de Denjoy-Carleman $\mathcal{C}_M$**. Ce sont des classes de fonctions C^∞ déterminées par une suite $M = (M_n)$ de bornes sur leurs dérivés[1]. Leur étude remonte à E. Borel et Hadamard [Bo1900], [Ha1923], autour des questions concernant les EDP linéaires. Denjoy et Carleman ont caractérisé les conditions nécessaires sur la suite M pour que la classe soit quasi-analytique [Denj1921], [Car1926].

(II) **Les fonctions C^∞-définissables**. Une collection d'ensembles $\mathcal{S} = (\mathcal{S}_n)_{n\in\mathbb{N}}$, où $\mathcal{S}_n$ sont des ensembles de sous-ensembles de $\mathbb{R}^n$, est une structure *o-minimale* si, en substance, elle est fermée relativement aux opérations booléennes usuelles, le produit cartésien, les projections suivant une coordonnée, et si tous les ensembles dans $\mathcal{S}_1$ ont un nombre fini de composantes connexes. Une fonction $f : U \subset \mathbb{R}^n \to \mathbb{R}$ est dite *définissable* dans une structure $\mathcal{S}$ si son graphe est un ensemble dans $\mathcal{S}$. Les classes de fonctions C^∞-définissables dans une structure o-minimale polynomialement borné $\mathcal{S}$ définissent des classes quasi-analytiques $\mathcal{Q}$, selon un résultat de C. Miller [Mi95].

Si l'intérêt pour les classes quasi-analytiques remonte au début du XXe siècle, leurs propriétés algébriques et géométriques sont loin d'être comprises. Du point de vue de la théorie des modèles, leur structure est comparable à celle des fonctions analytiques : elles sont o-minimales polynomialement bornées et modèle complète [RSW03]. Néanmoins, la plupart des propriétés algébriques des fonctions analytiques ne s'étendent pas aux classes quasi-analytique ; voir §3.7. Par exemple, il n'y a pas de Théorème de Préparation de Weierstrass quasi-analytique [Chi76], [ABBNZ14], [PR13], et plusieurs questions importantes concernant la division, la factorisation,

[1] Malheureusement, la notation classique de la suite est M, qui est utilisée dans ce mémoire également pour les variétés lisses.

la composition, etc, restent ouvertes. En particulier, on ne sait même pas si tous les anneaux locaux des classes quasi-analytiques sont noethériens.

Dans mes premiers travaux sur le sujet, avec Bierstone et Biborski [BdSBB17] et avec Bierstone et Chow [BdSBC18], nous avons combiné des techniques de résolution des singularités avec le *prolongement quasi-analytique*, une technique qui, à notre connaissance, est nouvelle dans la littérature; voir §5.6. Essentiellement, cette technique est utilisée pour "propager" sur un voisinage des propriétés vérifiées formellement (sans utiliser de propriétés algébriques telles que la platitude fidèle). En particulier, les résultats antérieurs sur le sujet (e.g. [Now13], [ChCh99], [ChCh01]) demandaient d'avoir des solutions formelles *partout*, pas seulement en un seul point. Cette méthode nous a permis d'étudier plusieurs problèmes concernant les solutions des équations quasi-analytiques, voir §5.2, et la composition des fonctions quasi-analytiques, voir §5.3. D'autres questions importantes sur les fonctions quasi-analytiques, concernant la division, la factorisation, la préparation de Weierstrass, etc., entrent dans le cadre de ces travaux (ou sont étroitement liées) et sont également discutées. Très récemment, en collaboration avec Bierstone et Kiro [BdSBK20], nous avons commencé à étudier les limites de la technique de résolution des singularités en géométrie quasi-analytique, voir §5.4.

Récemment, en collaboration avec Bierstone [BdSB19], nous avons combiné nos techniques précédentes avec des techniques de *résolution des singularités adaptées aux log-dérivées*, voir §1.2 et §4.6, afin de prouver une monomialisation des morphismes dans ces catégories, voir 4.4. Cela est un nouvel outil algébrique en géométrie quasi-analytique, qui nous permet, par exemple, de démontrer la rectiliniarisation des ensembles définissables et l'élimination des quantificateurs pour certaines structures o-minimales, sans avoir besoin des arguments de la théorie des modèles, voir §5.5.

Les préliminaires sur les classes quasi analytiques se trouvent dans le chapitre 3. Dans le chapitre 5 je donne un aperçu de nous résultats, et je présente les idées principales derrière la technique de prolongement quasi-analytique.

1.4 Géométrie sous-Riemannienne et la Conjecture de Sard

Dans ce qui suit, je présente la conjecture de Sard en termes de trajectoires (éventuellement, seulement localement définies) d'un système de contrôle (voir [Ag14, BdSR18, Mo02, Ri14] pour une présentation plus générale). Soit M une variété régulière de dimension $m \geq 3$ et Δ une distribution non-holonome de rang $k < m$, c'est-à-dire que Δ est un sous-fibré vectoriel régulier de TM tel que la condition de Hörmander $Lie\{\Delta\} = TM$ est satisfaite, i.e. l'algèbre de Lie engendrée par les sections de Δ est égal à TM. Soient $X^1, \ldots, X^k$ des champs de vecteurs qui engendrent Δ et soit $x \in M$ un point fixé. Il existe un ouvert maximal $\mathcal{U}^x \subset L^2([0,1], \mathbb{R}^k)$ tel que la solution du problème de Cauchy :

$$\dot{x}(t) = \sum_{i=1}^{k} u_i(t) X^i(x(t)), \quad x(0) = x, \text{ et } u \in \mathcal{U}^x \tag{1.1}$$

soit bien définie. On considère l'application "end-point mapping" :

$$\begin{array}{rccc} E^x : & \mathcal{U}_x & \to & \mathbb{R}^n \\ & u & \mapsto & \varphi(x,1,u) \end{array}$$

où $\varphi(x,t,u)$ est une solution de (1.1). On dira que le contrôle $u \in \mathcal{U}^x$ est singulier (par rapport à x) si E^x n'est pas une submersion en u, et on note par $\mathcal{S}^x$ l'ensemble des contrôles singuliers (par rapport à x). On considère :

$$\mathcal{X}^x := E^x(\mathcal{S}^x) \subset M$$

L'ensemble $\mathcal{X}^x$ est l'ensemble des valeurs critiques de l'application E^x et, en analogie avec le Théorème de Sard, on peut conjecturer que $\mathcal{X}^x$ **est de mesure de Lebesgue nulle**. Ceci est précisément l'énoncé de la *conjecture de Sard* quand $\dim(M) > 3$ (nous renvoyons le lecteur à [Mo02, §10.2] pour un aperçu historique). Malheureusement, le théorème de Sard n'est pas valable en dimension infinie. D'après Montgomery "A positive answer [to the Sard Conjecture] would lead to a fundamental progress in understanding the structure of geodesics" [Mo02, page 140]. Résoudre cette conjecture serait utile, par exemple, pour des questions de type Monge-Ampère en géométrie sous-Riemannienne car cela permettrait d'utiliser des arguments de transport (c.f. [Ri14, Section 3.6]). En dehors de quelques résultats sous de fortes hypothèses, e.g. [DMOPV16], la conjecture reste largement ouverte lorsque $dim\, M \geq 4$.

Si $dim\, M = 3$, l'ensemble des valeurs critiques $\mathcal{X}^x$ de l'application "end-point mapping" E^x, est contenu dans une surface $\Sigma \subset M$ qu'on appelle surface de Martinet. Dans ce cas, la conjecture de Sard affirme que **tous les ensembles $\mathcal{X}^x$ ont mesure de Hausdorff 2-dimensionnelle nulle**. La validité de cette conjecture est étayée par une importante contribution faite par Zelenko et Zhitomirskii [ZZ95] dans les années 90. Ils démontrent la conjecture sous deux hypothèses supplémentaires : (i) la surface Σ est lisse et (ii) la distribution Δ est générique (pour la topologie $\mathcal{C}^\infty$ de Whitney).

En collaboration avec L. Rifford [BdSR18], nous démontrons la Conjecture de Sard en dimension trois, sous l'unique hypothèse (i). Les preuves de nos résultats reposent sur un mélange d'arguments de topologie différentielle, de théorie géométrique de la mesure et de résolution des singularités (c.f. §1.2). Plus tard, en collaboration avec A. Figalli, A. Parusiński et L. Rifford [BdSFPR18], nous avons amélioré nos techniques précédentes en les combinant avec des arguments de géométrie symplectique et des résultats sur la régularité des transitions de Poincaré, afin de prouver la (version forte de la) Conjecture de Sard pour les variétés analytiques tridimensionnelles (sans aucune hypothèse supplémentaire). En combinant notre résultat avec le travail de E. Hakavuori et E. Le Donne [HL16], nous avons montré que, sous la même hypothèse, toutes les géodésiques sous-riemanniennes sont C^1. Une partie de nos techniques se généralise facilement à des dimensions plus élevées, tandis que les techniques de résolution des singularités doivent être approfondies, c.f. §1.2.

Dans le chapitre 6, je donne un aperçu de ces résultats, et je présente les principales idées derrière nos preuves.

1.5 Perspectives

Dans cette partie nous expliquons très brièvement plusieurs nouveaux projets liés aux travaux présentés dans ce mémoire. Les projets sont répartis selon les trois directions principales du mémoire, bien que certaines soient étroitement liées.

Résolution des singularités et log-dérivées J'ai actuellement plusieurs projets liés à la résolution des singularités et les log-dérivées. De nouvelles contributions ont suscité l'enthousiasme de la communauté de géométrie algébrique pour ce domaine [ATW20a, ATW20b, BdSB19, McQM19]. En utilisant les éclatements de Kummer, Abramovich, Temkin et Włodarczyk ont démontré une principalisation des idéaux relative à des morphismes log-réguliers [ATW20a] [ATW20b], et ils ont utilisé leur résultat pour obtenir une version de la réduction des morphismes. Dans nos travaux avec Bierstone [BdSB19], nous prouvons la monomialisation locale des morphismes quasi-analytiques via des éclatements lisses, en nous appuyant sur nos techniques de résolution de singularités adaptées aux log-dérivées. Enfin, en utilisant des éclatements pondérés, McQuilan et Marzo ont fourni une nouvelle méthode de résolution des singularités des variétés [McQM19], qui suit la ligne générale utilisée par Panazzolo dans ses travaux sur la réduction des champs vectoriels [Pan06]. Dans tous ces travaux, le comportement asymétrique des dérivations sous éclatements est traité de manières différentes (bien qu'il semble exister des similitudes). J'ai l'intention d'étudier et de combiner ces techniques dans mes futurs projets.

Plus concrètement, voici deux projets en cours. Premièrement, je travaille avec Fantini et Pichon sur un projet concernant l'étude de la métrique interne et la configuration polaire des variétés singulières [BdSFP19] et [BdSFP20]. Nous utilisons la résolution des singularités en familles (suite aux travaux de Teissier), les projections génériques et la topologie. Avec Pichon, nous co-encadrons un doctorant qui travaille actuellement sur ce sujet. Je crois que mes travaux sur les coordonnées de Hsiang-Pati (obtenus en collaboration avec Bierstone, Grandjean et Milman, voir §4.2) pourraient être importants pour traiter les dimensions supérieures à 2. Deuxièmement, je travaille actuellement avec Grieser sur l'étude des géodésiques des surfaces singulières. J'ai l'intention de mélanger nos métho-des de résolution des singularités des métriques [BdSBGM17], voir §4.2, avec l'approche que nous avons utilisée pour traiter la conjecture de Sard dans [BdSFPR18], voir §6.9.

Géométrie réelle quasi-analytique Les solutions de problèmes concernant les classes Denjoy-Carleman C_M utilisant la résolution des singularités, conduisent en général à une *perte de régularité* (c.f. Théorèmes 5.2.1, 5.2.2, 5.3.1 et 5.3.2). En collaboration avec Bierstone et Kiro, nous étudions la question de savoir si la perte de régularité est une conséquence de l'utilisation de la technique de la résolution des singularités, ou si elle est intrinsèque aux questions sur les classes de Denjoy-Carleman. Nous avons récemment accompli la première étape de notre projet [BdSBK20], et nous allons étudier le problème de la perte de régularité sous division, via des techniques d'analyse complexe et d'équations différentielles. Notons que, s'il s'avérait que la perte de régularité est inévitable lors de la division des fonctions dans les classes Denjoy-Carleman, cela impliquerait que les anneaux locaux de fonctions quasi-analytiques de Denjoy-Carleman ne sont pas noethériens.

Géométrie sous-Riemannienne et la Conjecture de Sard En collaboration avec Parusiński et Rifford, nous travaillons actuellement sur la conjecture de Sard dans des dimensions plus grandes. La première étape de notre projet consiste à généraliser certaines des idées de [BdSFPR18] à des dimensions supérieures, en s'appuyant sur la géométrie sous-analytique et symplectique, pour établir un nouveau cadre de travail qui soit bien adapté à l'étude de la conjecture de Sard. La deuxième étape sera de faire une étude systématique du comportement transcendantal des champs de vecteurs caractéristiques. Il est trop tôt pour spéculer sur les techniques qui pourraient interférer dans la deuxième étape, mais je voudrais rappeler que les feuilletages par courbes caractéristiques doivent satisfaire une contrainte différentielle importante, voir §6.7. Étant donnés les progrès récents en résolution des singularités et log-dérivées, j'ai l'intention d'étudier s'il est possible d'obtenir une résolution des singularités pour ces feuilletages.

Chapter 2

Listes des travaux

Voici la liste de mes travaux avec les références à la liste bibliographique du mémoire

2.1 Listes des travaux présentés pour l'habilitation

1. [BdSBK20] A. Belotto da Silva, E. Bierstone and A. Kiro *Sharp Estimates for Blowing Down Functions in a Denjoy-Carleman Class*, to appear in IMRN.
2. [BdSB19] A. Belotto da Silva, E. Bierstone, *Monomialization of a quasianalytic morphism*, preprint, arXiv 1907.09502 [math.AG], (2019).
3. [BdSFPR18] A. Belotto da Silva, A. Figalli, A. Parusiński et L. Rifford *Strong Sard Conjecture and regularity of singular minimizing geodesics for analytic sub-Riemannian structures in dimension 3*, preprint, hal-01889705, (2018).
4. [BdSR18] A. Belotto da Silva et L. Rifford *The sub-Riemannian Sard conjecture on Martinet surfaces* Duke Mathematical Journal, Volume 167, Number 8 (2018), 1433-1471.
5. [BdSBC18] A. Belotto da Silva, E. Bierstone and M. Chow *Composite quasianalytic functions* Compositio Mathematica, Volume 154, Issue 9 (2018) pp. 1960-1973.
6. [BdS18] A. Belotto da Silva *Local monomialization of a system of first integrals of Darboux type* Revista Matemática Iberoamericana, Volume 34, Issue 3, (2018), pp. 967–1000.

7. [BdSB17] A. Belotto da Silva et E. Bierstone *Resolution of singularities of differential forms and Hsiang-Pati coordinates* Advanced Lectures in Mathematics, Volume 39: Hodge theory and L2 analysis, edited by Lizhen Ji, International Press, (2017), 127-150 pp.

8. [BdSBB17] A. Belotto da Silva, I. Biborski and E. Bierstone *Solutions of quasianalytic equations* Selecta Mathematica, Volume 23, Number 4, (2017), 2523-2552 pp.

9. [BdSBGM17] A. Belotto da Silva, E. Bierstone, V. Grandjean and P. Milman *Resolution of singularities of the cotangent sheaf of a singular variety*, Advances in Mathematics **307**, (2017), 780–832 pp.

10. [BdS16b] A. Belotto da Silva *Local Resolution of Ideals Subordinated to a Foliation* RACSAM **110** (2016), no. [2], 841-862 pp.

2.2 Liste complète des travaux par thèmes

I - Résolution des singularités adaptées aux log-dérivées :

- [BdSP19] A. Belotto da Silva and D. Panazzolo *Generalized Flow-Box property for singular foliations*, RACSAM, (113), pages 3949-3965, 2019.

- [BdS18] A. Belotto da Silva *Local monomialization of a system of first integrals of Darboux type* Revista Matemática Iberoamericana, Volume 34, Issue 3, (2018), pp. 967–1000.

- [BdSB17] A. Belotto da Silva et E. Bierstone *Resolution of singularities of differential forms and Hsiang-Pati coordinates* Advanced Lectures in Mathematics, Volume 39: Hodge theory and L2 analysis, edited by Lizhen Ji, International Press, (2017), 127-150 pp.

- [BdSBGM17] A. Belotto da Silva, E. Bierstone, V. Grandjean and P. Milman *Resolution of singularities of the cotangent sheaf of a singular variety*, Advances in Mathematics **307**, (2017), 780–832 pp.

- [BdS16b] A. Belotto da Silva *Local Resolution of Ideals Subordinated to a Foliation* RACSAM **110** (2016), no. [2], 841-862 pp.

- [BdS16a] A. Belotto *Global resolution of singularities subordinated to a 1-dimensional foliation* Journal of Algebra **447** (2016), 397-423 pp.

II - Géométrie réelle quasi-analytique

- [BdSBK20] A. Belotto da Silva, E. Bierstone and A. Kiro *Sharp Estimates for Blowing Down Functions in a Denjoy-Carleman Class*, to appear in IMRN.
- [BdSBC18] A. Belotto da Silva, E. Bierstone and M. Chow *Composite quasianalytic functions* Compositio Mathematica, Volume 154, Issue 9 (2018) pp. 1960-1973.
- [BdSBB17] A. Belotto da Silva, I. Biborski and E. Bierstone *Solutions of quasianalytic equations* Selecta Mathematica, Volume 23, Number 4, (2017), 2523-2552 pp.

I et II - Rés. des sing. adaptées aux log-dérivées et géométrie quasi-analytique :

- [BdSB19] A. Belotto da Silva, E. Bierstone, *Monomialization of a quasianalytic morphism*, preprint, arXiv 1907.09502 [math.AG], 2019.

III - Géométrie sous-Riemannienne et Conjecture de Sard

- [BdSFPR18] A. Belotto da Silva, A. Figalli, A. Parusiński et L. Rifford *Strong Sard Conjecture and regularity of singular minimizing geodesics for analytic sub-Riemannian structures in dimension 3*, preprint, hal-01889705, 2018.
- [BdSR18] A. Belotto da Silva et L. Rifford *The sub-Riemannian Sard conjecture on Martinet surfaces* Duke Mathematical Journal, Volume 167, Number 8 (2018), 1433-1471.

IV - Géométrie analytique locale

- [BdSCR20] A. Belotto da Silva, O. Curmi et G. Rond, *A proof of Gabrielov's rank Theorem*, preprint, arXiv, 2008.13130 [math.CV] 2020.
- [BdSFP20] A. Belotto da Silva, L. Fantini et A. Pichon *Lipschitz normal embeddings and polar exploration of complex surface germs*, preprint, arXiv, 2006.01773 [math.AG], 2020.
- [BdSFP19] A. Belotto da Silva, L. Fantini et A. Pichon *Inner Lipschitz geometry of complex surfaces: a valuative approach*, 1905.01677 [math.AG], 2019.

V - Optimisation et Machine Learning

- [BdSG20] A. Belotto da Silva et M. Gazeau, *A general system of differential equations to model first order adaptive algorithms*, Journal of Machine Learning Research, 21(129):1-42, 2020.

VI - Théorie qualitative des ODE

- [BdSE19] A. Belotto da Silva and J. G. Espín Buendía *Topological classification of limit periodic sets of polynomial planar vector fields*, Publicacions Matemàtiques, Volume 63, Number 1 (2019), 105-123pp.
- [BdS16] A. Belotto *Examples of infinitesimal non-trivial accumulation of secants in dimension three* J. of Dynamical and Control Systems **22** (2016), no [1], 45–54.
- [BdS12] A. Belotto *Analytic varieties as limit periodic sets* Qualitative Theory of Dynamical Systems **11** (2012), no [2], 449–465 pp.

Chapter 3

Preliminary: resolution of singularities, singular foliations and quasianalytic classes

We have chosen to use the language of complex and real analytic geometry throughout this chapter (up to Section 3.6). Some notions are introduced in a specialized case instead of in full generality, in order to keep the presentation as elementary as possible. Note that all notions used below can be extended to the algebraic and quasianalytic categories, possibly under extra technical assumptions.

3.1 Analytic spaces and ideal sheaves

Let M be a complex or real-analytic (smooth) manifold. We denote by $\mathcal{O}_M$ the sheaf of analytic functions over M. Note that $(M, \mathcal{O}_M)$ is a ringed space. Given a point $\mathfrak{a} \in M$, we denote by $\mathcal{O}_{\mathfrak{a}}$ the localization of $\mathcal{O}_M$ at $\mathfrak{a}$, that is, the ring of analytic function germs at $\mathfrak{a}$. We recall that $\mathcal{O}_{\mathfrak{a}}$ is a local ring. Denote its maximal ideal by $m_{\mathfrak{a}}$. At each point, there exists an analytic coordinate system $\boldsymbol{x} = (x_1, \ldots, x_n)$ centered at $\mathfrak{a}$ such that $\mathcal{O}_{\mathfrak{a}}$ is isomorphic to $\mathbb{K}\{x\}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

An ideal sheaf $\mathcal{I}$ of $\mathcal{O}_M$ is a sub-sheaf whose stalks $\mathcal{I}_{\mathfrak{a}} = \mathcal{I} \cdot \mathcal{O}_{\mathfrak{a}}$ are ideals of $\mathcal{O}_{\mathfrak{a}}$ for every $\mathfrak{a} \in M$. We say that $\mathcal{I}$ is everywhere non-zero if $\mathcal{I}_{\mathfrak{a}} \neq (0)$ for every $\mathfrak{a} \in M$. We say that $\mathcal{I}$ is of finite type if at every point $\mathfrak{a} \in M$, there exists a neighborhood U of $\mathfrak{a}$ and a finite number of analytic functions $f_1, \ldots, f_n : U \to \mathbb{K}$ such that $\mathcal{I}_{\mathfrak{b}}$ is generated by the germs of $f_1, \ldots, f_n$ at every point $\mathfrak{b} \in U$. We say that $\mathcal{I}$ is coherent if it is of finite type and, in the above notation, the kernel of the morphism $(f_1, \ldots, f_n) : U \to \mathbb{K}^n$ is a module of finite type. We recall that, when the ambient space M is smooth, the structural ring $\mathcal{O}_M$ is coherent and a finite type ideal sheaf $\mathcal{I}$ is always coherent. Coherence is a desirable property in most theories in algebraic geometry and is going to be an over-arching hypothesis in the algebraic and analytic category[1]. This hypothesis is arguably very mild when we study a complex analytic sets $X \subset M$ because, according to Oka's Theorem, the sheaf generated by all germs of $\mathcal{O}_M$ which are zero along X is coherent. In the real-analytic case, nevertheless, coherence is a non-trivial hypothesis; we refer the interested reader to [Na66] for several examples of real analytic sets which admit no possible (non-constant) coherent sheaf.

The zero set of a coherent ideal sheaf $\mathcal{I}$ is the set:

$$Z(\mathcal{I}) = |X| = \{\mathfrak{a} \in M;\ f(\mathfrak{a}) = 0, \forall f \in \mathcal{I}_{\mathfrak{a}}\}$$

If we set $\mathcal{O}_X = \mathcal{O}_M/\mathcal{I}$, then $X = (|X|, \mathcal{O}_X)$ is a ringed space, and we call it a sub-space of M. More generally, a coherent analytic space is, roughly, a ringed $X = (|X|, \mathcal{O}_X)$ where $\mathcal{O}_X$ is coherent and which admits everywhere locally defined embedding $X|_U \hookrightarrow M$ into a smooth variety M (where U denotes an open subset of X) where the image of X is as above.

The singular set of an ideal sheaf $\mathcal{I}$ is given by:

$$\mathrm{Sing}(\mathcal{I}) = \mathrm{Sing}(X) = \{\mathfrak{a} \in Z(\mathcal{I});\ \mathcal{O}_{\mathfrak{a}}/\mathcal{I}_{\mathfrak{a}} \text{ is not regular}\}.$$

If $\mathrm{Sing}(\mathcal{I}) = \emptyset$, we say that $\mathcal{I}$ is smooth. We can extend these definitions to analytic spaces X, in the obvious way. In the real case, note that the definition of singular set of a space does not coincide with the set of points where $|X|$ is not a manifold; for example is we consider the Whitney umbrella $|X| = (x^2 - yz^2 = 0)$ and $\mathcal{O}_X = \mathcal{O}_{\mathbb{R}^3}/(x^2 - yz^2)$, then $|X| \setminus \{x = z = 0, y \geq 0\}$ is everywhere locally a manifold (although, not of constant dimension) while $\mathrm{Sing}(X) = \{x = z = 0\}$.

Let I be an ideal in a commutative ring R. We say that I is *reduced* if it coincides with its radical $\sqrt{I} = \{f \in R;\ \exists a \in \mathbb{N}, f^a \in I\}$. An ideal sheaf is said to be *reduced* (respectively *principal*) if its stalks $\mathcal{I}_{\mathfrak{a}}$ are reduced (respectively *principal*) for every $\mathfrak{a} \in Z(\mathcal{I})$; we extend the notion of being reduced to analytic spaces X in the obvious way. In the complex case, the hypothesis that $\mathcal{I}$ is reduced guarantees that $\mathrm{Sing}(\mathcal{I}) \neq Z(\mathcal{I})$ (respectively, $\mathrm{Sing}(X) \neq X$), but the same does not hold in the real case. For example, if we consider $\mathcal{I} = (x^2 + y^2)$ in $\mathbb{R}^2_{x,y}$, then $Z(\mathcal{I}) = (0, 0) = \mathrm{Sing}(\mathcal{I})$.

[1] In the quasianalytic category, coherence is too strong of a hypothesis even locally (because local rings are not known to be Noetherien), and we will work with the notion of privileged sheaves instead, see Definition 3.6.3.

Remark 3.1.1. *If the ideal sheaf $\mathcal{I}$ is principal and reduced, then the singular set Sing($\mathcal{I}$) has, at least, codimension two. This remarks plays an important role when we discuss the Martinet surface in sub-Riemannien geometry, see Remark 6.2.3 below.*

3.2 Resolution of singularities

The following presentation is independent of the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. The reader may find details of the results described below, with proofs, in [BM08], see also [BM97, W09].

Simple normal crossing divisors: A finite simple normal crossing divisor E over M is an ordered collection $E = (E^{(1)}, ..., E^{(l)})$, where each $E^{(i)}$ is a smooth divisor on M such that $\sum_i E^{(i)}$ is a reduced divisor with simple normal crossings, that is, at every point $\mathfrak{a} \in E$, there exists an adapted coordinate system $(\boldsymbol{u}, \boldsymbol{v}) = (u_1, \ldots, u_r, v_1, \ldots, v_s)$, with $r + s = m = \dim(M)$, centered at $\mathfrak{a}$ such that $E = (u_1 \cdots u_r = 0)$ locally at $\mathfrak{a}$. We say that the coordinate system $(\boldsymbol{u}, \boldsymbol{v})$ is *adapted* to E

A *simple normal crossing* (SNC) divisor E over M is a locally compact set $E \subset M$ whose restriction to any relatively compact open subset $U \subset M$ is a finite simple normal crossing divisor. In the algebraic category, every SNC divisor is a finite SNC divisor. In what follows, we consider pairs (M, E), where E is a simple normal crossing divisor.

Remark 3.2.1. *Throughout this work, all divisors are reduced, that is, we don't consider sums $\sum_i a_i E^{(i)}$ with $a_i \neq 1$.*

Blowing-up (with smooth centers) in $\mathbb{K}^n$. We start by defining a blowing-up (with smooth center) over $\mathbb{K}^n$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Consider a coordinate system $(x_1, \ldots, x_n)$ of $\mathbb{K}^n$. A blowing-up of $\mathbb{K}^n$ with center $\mathcal{C} = Z(x_1, \ldots, x_t)$, is the analytic projection:

$$\sigma : \widetilde{M} \subset \mathbb{K}^n \times \mathbb{P}^{t-1}_{\mathbb{K}} \to \mathbb{K}^n .$$

where:

$$\widetilde{M} = \left\{ (x, y) \in \mathbb{C}^n \times \mathbb{P}^{t-1} \ : \ x_i y_j = y_i x_j \, , \ 1 \leq i, j \leq t \right\} .$$

The set $\widetilde{M}$ will be called the blown-up space, $\mathcal{C}$ will be called the center of blowing-up, and $E := \sigma^{-1}(\mathcal{C})$ will be called the exceptional divisor of σ. Note that σ is an analytic isomorphism from $\widetilde{M} \setminus E$ to $\mathbb{K}^n \setminus \mathcal{C}$.

Remark 3.2.2. *If the center of the blowing-up σ is a hypersurface, say $(x_1 = 0)$, then σ is an isomorphism. Even in this case, we call E the exceptional divisor. This choice conflicts with the usual notion of exceptional divisor of a map in algebraic geometry, which represents the locus where a map is not locally analytically an isomorphism. In any case, this choice is, arguably, the correct notion for resolution of singularities, c.f. [Kol07, Warning 3.20(2)].*

Directional charts. In the notation of the previous paragraph, we can cover $\widetilde{M}$ via t local charts. More precisely, there exists open sets $U_k \subset \widetilde{M}$ with $k = 1, \dots, t$, which cover $\widetilde{M}$, and coordinate systems $(u, \boldsymbol{v}) = (u, v_1, \dots, v_{k-1}, v_{k+1}, \dots, v_n)$ over U_k such that $\sigma_k := \sigma|_{U_k}$ is given by:

$$\sigma_k : \begin{cases} x_i = u \cdot v_i & \text{if } i = 1, \dots, k-1, k+1, \dots, t \\ x_k = u & \\ x_i = v_i & \text{if } i = t+1, \dots, n. \end{cases}$$

Note that $E = (u = 0)$, implying that $(u, \boldsymbol{v})$ is an adapted coordinate system. The map $\sigma_k : U_k \to \mathbb{K}^n$ is called the x_k-chart of the blowing-up (intuitively, this name captures the fact that the pull-back of x_k by σ_k is a generator of E).

Blowing-up in general manifolds. Given a coherent and smooth ideal sheaf $\mathcal{I}_{\mathcal{C}}$ of $\mathcal{O}_M$ (that is $\mathrm{Sing}(\mathcal{I}_{\mathcal{C}}) = \emptyset$), we can define the blowing-up $\sigma : \widetilde{M} \to M$ with *center* $\mathcal{C} = Z(\mathcal{I}_{\mathcal{C}})$ by patching the local construction of the previous paragraphs. Indeed, it is possible to define a blowing-up with center given by any coherent ideal sheaf $\mathcal{I}$ (not necessary smooth). We omit the details in here, because all blowings-up of this work are smooth (that is, the center is smooth).

Let $\sigma : \widetilde{M} \to M$ be a blowing-up with center $\mathcal{C}$ and exceptional divisor F, and X be a subset of M. There are two possible transforms of X which are of interest:

$$\text{the total transform } X^* := \sigma^{-1}(X),$$
$$\text{the strict transform } X^{st} := \overline{\sigma^{-1}(X \setminus \mathcal{C})}.$$

Now, let E be a SNC divisor over M. We say that the blowing-up σ is *compatible* with E, if the center $\mathcal{C}$ has normal crossings with E, that is, at every point $\mathfrak{a} \in E$, there exists adapted coordinate system $(\boldsymbol{u}, \boldsymbol{v})$ such that $\mathcal{C}_{\mathfrak{a}} = (u_1, \dots, u_{r'}, v_1, \dots, v_{s'})$ with $0 \leq r' \leq r$ and $0 \leq s' \leq s$. In this case, the set $\widetilde{E} = E^{st} \cup F$ can be checked to be a SNC crossing divisors, and we denote the blowing-up by $\sigma : (\widetilde{M}, \widetilde{E}) \to (M, E)$.

Next, given a coherent ideal sheaf $\mathcal{I}$ over $\mathcal{O}_M$, the *total* transform[2] of $\mathcal{I}$ is an ideal sheaf $\widetilde{\mathcal{I}} := \sigma^*(\mathcal{I})$ whose stalk at a point $\mathfrak{b} \in \widetilde{M}$ is generated by:

$$f \circ \sigma; \quad \forall f \in \mathcal{I}_{\sigma(\mathfrak{b})}$$

In particular, we denote by $\mathcal{I}_F := \sigma^*(\mathcal{C})$, which is the coherent (and reduced) ideal sheaf whose zero locus is the exceptional divisor F.

Resolution of singularities of ideal sheaves. Consider a pair (M, E), where E is a SNC divisor, and a coherent ideal sheaf $\mathcal{I}$ of $\mathcal{O}_M$. We say that $\mathcal{I}$ is a *monomial* ideal sheaf (relative to (M, E)) if $\mathcal{I}$ is a principal ideal sheaf with zero locus contained in E such that[3], at every point $\mathfrak{a} \in E$, there exists an adapted coordinate system $(\boldsymbol{u}, \boldsymbol{v})$ such that

$$\mathcal{I}_{\mathfrak{a}} = (\boldsymbol{u}^{\boldsymbol{\alpha}}) = (u_1^{\alpha_1} \cdots u_r^{\alpha_r})$$

with $\boldsymbol{\alpha} \in \mathbb{N}^r$. The objective of a resolution of singularities is to obtain a finite number of blowings-up (in every relatively open set) which monomialize the ideal sheaf $\mathcal{I}$.

Theorem 3.2.3 (Monomialisation of ideal sheaves [Hi64, AHV75a, AHV75b] ; see a modern presentation in [BM08])**.** *Let M be a complex- or real-analytic manifold, E a SNC divisor, and $\mathcal{I}$ a coherent and everywhere non-zero ideal sheaf. For every relatively compact open set $M_0 \subset M$, denote by $E_0 = E \cap M_0$ and $\mathcal{I}_0$ the restriction of $\mathcal{I}$ over M_0. There exists a sequence of compatible blowings-up with smooth centers*

$$(M_t, E_t) \xrightarrow{\sigma_t} \ldots \xrightarrow{\sigma_2} (M_1, E_1) \xrightarrow{\sigma_1} (M_0, E_0)$$

such that, denoting by $\sigma_0 := \sigma_1 \circ \cdots \circ \sigma_t$, the pull-back $\mathcal{I}_t := \sigma_0^*(\mathcal{I}_0)$ is monomial (relative to (M_t, E_t)).

Furthermore, there exists a bi-meromorphic morphism $\sigma : (\widetilde{M}, \widetilde{E}) \to (M, E)$, such that, for every relatively compact open set M_0, we have that $\sigma|_{M_0}$ is of the previous kind.

Remark 3.2.4. *In the above result, we can attach the morphism σ to the triple $(M, \mathcal{I}, E)$ in a way that is functorial relative to smooth morphisms, see [Kol07, §3.4]. We won't introduce this notion in this memoir, although it appears in the works presented in it.*

[2]There are also notions of *weak* and *strict* transforms for ideal sheaves. We do not need to define them in this memoir, although they appear in the works which it presents

[3]If the field is algebraically closed, such as $\mathbb{C}$, then every a principal ideal sheaf with zero locus contained in E is monomial. This is not true in the real case, nevertheless, e.g. $\mathcal{I} = (u_1^2 + u_2^2)$ and $E = (u_1 u_2 = 0)$.

Resolution of singularities of an analytic space: Let $X = (|X|, \mathcal{O}_X)$ be a coherent (and reduced) complex or real-analytic space. We say that X is embedded in a manifold M if there exists a coherent (and reduced) ideal sheaf $\mathcal{I}_X$ of $\mathcal{O}_M$ whose zero locus is $|X|$ (as a set) and $\mathcal{O}_X = \mathcal{O}_M/\mathcal{I}_X$. It follows, in an easy way, that a monomialization of $\mathcal{I}_X$ yields an embedded resolution of singularities of X, that is:

Theorem 3.2.5 (Embedded resolution of singularities [Hi64, AHV75a, AHV75b] ; see a modern presentation in [BM08])**.** *Let $X = (|X|, \mathcal{O}_X)$ be a coherent (and reduced) complex or real-analytic space embedded in a manifold M, and let E be a SNC divisor over M. For every relatively compact open set $M_0 \subset M$, denote by $E_0 = E \cap M_0$ and $X_0 = X \cap M_0$. There exists a sequence of compatible blowings-up with smooth centers*

$$(M_t, E_t) \xrightarrow{\sigma_t} \dots \xrightarrow{\sigma_2} (M_1, E_1) \xrightarrow{\sigma_1} (M_0, E_0)$$

such that, denoting by $\sigma_0 := \sigma_1 \circ \cdots \circ \sigma_t$, the strict transform X_t of X_0 is a smooth analytic manifold. Furthermore, the restriction (X_t, F_t), given by $F_t = E_t \cap X_t$, is a SNC divisor.

The above result may be also made functorial in respect to smooth morphisms and, in particular, in respect to the restriction to open sets [BM08]. Recalling that an analytic space X is everywhere locally embedded (but not necessarily globally embedded), we obtain:

Theorem 3.2.6 (Resolution of singularities [Hi64, AHV75a, AHV75b]; see a modern presentation in [BM08])**.** *Let $X = (|X|, \mathcal{O}_X)$ be a coherent (and reduced) complex or real-analytic space. There exists a proper bi-meromorphic map*

$$\sigma : (X, F) \to (X_0, Sing(X_0))$$

where X is a smooth analytic space, F is a SNC divisor and σ is an isomorphism from $X \setminus F$ to $X \setminus Sing(X)$.

In the above Theorem, we can add the extra condition that σ coincides (whenever it is restricted to a relatively compact set) with a finite sequence of blowings-up with smooth centers in X. We note that this result does not follow in an easy way from resolution of ideal sheaves, because a smooth center in the ambient variety M, might not be smooth when restricted to X. All proofs that I am aware of use the notion of *presentation* of the multiplicity or the Hilbert Samuel function, c.f. [V89, BM97, EV03, V14].

3.3 Local resolution of singularities

We now introduce the main notions which are necessary in order to state our results of *local* resolution of singularities from [BdS16b, BdS18, BdSB19], see §4.3 and 4.4. Note that we do not introduce the notion of valuations (or étoiles of Hironaka), since we do not use these notions in our works.

Local blowings-up: A compatible *local* blowing-up $\tau : (\widetilde{U}, \widetilde{E}) \to (M, E)$ is a composite $\iota \circ \sigma$ of an inclusion of an open subset $\iota : \widetilde{U} \hookrightarrow \widetilde{M}$, and a (smooth) blowing-up $\sigma : (\widetilde{M}, \widetilde{E}) \to (M, E)$. In the algebraic case, an open subset is understood to mean with respect to the étale topology, so that a *local blowing-up* σ means the composite $\varepsilon \circ \sigma$ of an étale mapping $\iota : \widetilde{U} \hookrightarrow \widetilde{M}$ and a smooth blowing-up $\sigma : (\widetilde{M}, \widetilde{E}) \to (M, E)$.

Power substitution: Suppose that $\mathbb{K} = \mathbb{R}$. A (real) *power substitution* means, *roughly speaking*, a morphism $\rho : U' \to U$ of the form $u_i = \tilde{u}_i^{k_i}$, $i = 1, \ldots, r$, where each k_i is a positive integer, and U is a chart with coordinates $\boldsymbol{u} = (u_1, \ldots, u_r)$. In order to cover U, given ρ, we introduce the mapping

$$P : \coprod_{\boldsymbol{\epsilon}} U_{\boldsymbol{\epsilon}} \to U,$$

where $\coprod$ denotes disjoint union, $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_r) \in \{-1, 1\}^r$, each $U_{\boldsymbol{\epsilon}}$ is a copy of U', and $P_{\boldsymbol{\epsilon}} := P|_{U_{\boldsymbol{\epsilon}}}$ is given by $u_i = \epsilon_i \tilde{u}_i^{k_i}$, $i = 1, \ldots, r$. We assume that U is chosen in suitable symmetric product form, so that P is surjective. Power substitutions ρ can always be extended to more general *power substitutions* P defined in this way. A *local power substitution* $V' \to V$ is a composite $\iota \circ P$ of the inclusion of a suitable coordinate chart $\iota : U \hookrightarrow V$ and an extended power substitution $P : V' \to U$. Given a SNC divisor E over M, we say that a (local) power substitution $\rho : M' \to M$ is *compatible* with E if it is defined as above, with respect to adapted coordinate systems $(\boldsymbol{u}, \boldsymbol{v})$. In particular all the non-exceptional variables v are preserved.

Semi-proper covering: A family of morphisms $\{\sigma_\lambda : V_\lambda \to M\}$ is a *semi-proper covering* of M if $\{\sigma_\lambda\}$ is a covering of M (i.e., $\bigcup \sigma_\lambda(V_\lambda) = M$) and, for every compact subset K of M, there are finitely many compact $K_i \subset V_{\lambda(i)}$ such that $\bigcup_i \sigma_{\lambda(i)}(K_i) = K$.

3.4 Singular foliations

We follow [BB72]. Let Der_M denote the sheaf of analytic vector fields on M, i.e. the sheaf of analytic sections of TM. An *involutive singular distribution* is a coherent subsheaf Δ of Der_M such that, for each point p in M, the stalk $\Delta_{\mathfrak{a}} := \Delta \cdot \mathcal{O}_{\mathfrak{a}}$ is closed under the Lie bracket operation. Consider the quotient sheaf $Q = \mathrm{Der}_M/\Delta$. The *singular set* of Δ is defined by the closed analytic subset

$$\mathrm{Sing}(\Delta) = \{\mathfrak{a} \in M : Q_{\mathfrak{a}} \text{ is not a free } \mathcal{O}_{\mathfrak{a}} \text{ module}\}.$$

A singular distribution Δ is called *regular* if $\mathrm{Sing}(\Delta) = \emptyset$. On $M \setminus \mathrm{Sing}(\Delta)$ there exists an unique analytic subbundle L of $TM|_{M\setminus\mathrm{Sing}(\Delta)}$ such that Δ is the sheaf of analytic sections of L. We assume that the dimension of the $\mathbb{K}$ vector space L_p is the same for all points p in $M \setminus \mathrm{Sing}(\Delta)$ (this always holds if M is connected). It is called the *leaf dimension* of Δ and denoted by $\dim(\Delta)$. We say that Δ is an involutive *d-singular distribution* if $\dim(\Delta) = d$. An involutive singular distribution always generates a regular foliation outside its singular locus according to the Frobenius theorem, see e.g. [BCG91, §2, Theorem 1.1]; we refer the reader to [BCG91] for more details.

Consider a SNC divisor E over M. Let $Der_M(-logE)$ denote the coherent subsheaf of Der_M given by all derivations tangent to E, that is, for each point $\mathfrak{a} \in M$, a derivation ∂ is in $Der_M(-logE) \cdot \mathcal{O}_{\mathfrak{a}}$ if and only if $\partial[\mathcal{I}_E] \subset \mathcal{I}_E$, where $\mathcal{I}_E$ is the reduced ideal sheaf whose zero locus is E. A singular distribution Δ which is also a sub sheaf of $Der_M(-logE)$ is said to be tangent to E.

Blow-up of vector-fields in $\mathbb{K}^n$: Let us consider an analytic derivation ∂ defined on some open neighborhood $U \subset \mathbb{K}^n$ of 0, that is, a vector-field with expression:

$$\partial = \sum_{i=1}^{n} A_i(x)\partial_{x_i}, \quad \text{where} \quad A_i(x) \in \mathcal{O}_U, \quad i = 1, \ldots, n$$

and a blowing-up $\sigma : M \to \mathbb{K}^n$ with center $\mathcal{C} = Z(x_1, \ldots, x_t)$. Given a directional chart, say $\sigma_1 : U_1 \to \mathbb{K}^n$, we can define the *total transform* of the derivation ∂ in this chart as the meromorphic derivation ∂^* over $U_1 \cap \sigma^{-1}(U)$ which satisfies the following rule:

$$\partial^*(\sigma_1^*(f)) = \sigma_1^*[\partial(f)], \quad \forall\, f \in \mathcal{O}_U$$

It is now easy to compute the transform of each one of the derivations ∂_{x_i} in this directional chart explicitly:

$$\begin{aligned}
\partial_{x_1} &= \partial_u - \frac{1}{u}\sum_{i=2}^{t} v_i \partial_{v_i} \\
\partial_{x_i} &= \frac{1}{u}\partial_{v_i} && \text{for } i = 2, \dots, t \\
\partial_{x_i} &= \partial_{v_i} && \text{for } i = t+1, \dots, n
\end{aligned}$$

In general, it is not possible to define a total transform of the derivation ∂ over the entire manifold M. This provides a motivation to work with sheaves of derivations, instead with specific elements.

Blowing-up of a singular distributions Given a compatible blowing-up $\sigma : (\widetilde{M}, \widetilde{E}) \to (M, E)$ and a coherent involutive singular distribution $\Delta \subset Der_M$, note that the pull-back $\sigma^*(\Delta)$ may not be a sub-sheaf of $Der_{\widetilde{M}}$, because the pull-backs of vector fields are in general meromorphic (see previous paragraph). In this memoir, we make use of two types of transform of Δ which can be found in the literature:

- The *weak-transform* of Δ by σ, which we denote by $\sigma^w(\Delta) = \widetilde{\Delta}$, is the intersection of $\sigma^*(\Delta)$ (which is a sheaf of meromorphic derivations) with $Der_{\widetilde{M}}(-log\widetilde{E})$ (it follows from a result of Oka that the intersection is coherent).
- The *strict-transform* of Δ by σ, which we denote by $\sigma^{st}(\Delta) = \Delta^{st}$, is the the biggest sub-sheaf of $Der_{\widetilde{M}}(-log\widetilde{E})$ which coincides with $\widetilde{\Delta}$ over $\widetilde{M} \setminus \widetilde{E}$.

Note that we consider transforms which are tangent to $\widetilde{E}$. In particular, the singular locus of $\sigma^{st}(\Delta)$ might contain $\widetilde{E}$ (e.g. when the foliation is dicritical).

3.5 Reduction of singularities of planar line foliations

Consider an analytic derivation ∂ over an open and connected set $U \subset \mathbb{K}^n$, that is, a vector-field with expression:

$$\partial = \sum_{i=1}^{n} A_i(x)\partial_{x_i}, \quad \text{where} \quad A_i(x) \in \mathcal{O}_U, \quad i = 1, \dots, n$$

A point $\mathfrak{a} \in U$ is said to be a singularity of ∂ if $\partial(\mathfrak{a}) = 0$. We assume that $\partial \not\equiv 0$, which implies that the singular set $\mathrm{Sing}(\partial)$ is a proper analytic subset of U. We say that $\mathfrak{a}$ is an isolated singularity if there exists a neighborhood $U_{\mathfrak{a}}$ of $\mathfrak{a}$ where ∂ is non-singular everywhere over $U_{\mathfrak{a}} \setminus \mathfrak{a}$. We now recall the definition of elementary singularities (following [IY95, Definition 4.27]):

Definition 3.5.1 (Elementary singularities). *Suppose that $\mathfrak{a} \in U$ is a singular point of ∂ and consider the Jacobian matrix $Jac(\partial)$ associated to ∂, relative to local coordinate systems in a neighborhood of $\mathfrak{a}$, is given by:*

$$Jac(\partial) = \begin{bmatrix} \partial_{x_1} A_1(x) & \cdots & \partial_{x_n} A_1(x) \\ \vdots & \ddots & \vdots \\ \partial_{x_1} A_n(x) & \cdots & \partial_{x_n} A_n(x) \end{bmatrix}$$

We say that $\mathfrak{a}$ is an elementary singularity of ∂ if $Jac(\partial)$ evaluated at $\mathfrak{a}$ has at least one eigenvalue with non-zero real part.

Consider the sub-sheaf $\mathscr{L}$ of derivations Der_U generated by ∂, which is called a line-foliation. Note that the singular set of $\mathscr{L}$ coincides with the singular set of ∂. We extend the notion of elementary singularities to line foliations in the obvious way. The objective of a reduction[4] of singularities of a line foliation $\mathscr{L}$ is to provide a sequence of blowings-up so that the final strict transform of $\mathscr{L}$ only has elementary singularities. If $n = 2$, the classical Bendixson-Seidenberg Theorem (see e.g. [ADL06, Theorem 3.3] or [IY95, Theorem 8.14 and Section 8K] and references there-within) yields the following:

Theorem 3.5.2 (Reduction of singularities of planar line foliations). *Let $\mathscr{S}$ be an analytic surface, E be a SNC crossing divisor over $\mathscr{S}$, and $\mathscr{L}$ be an analytic line foliation over $\mathscr{S}$ which is everywhere non-zero (that is, at every point $\mathfrak{a} \in \mathscr{S}$, the stalk $\mathscr{L}_{\mathfrak{a}}$ is generated by a vector-field which is not identically zero in a neighborhood of $\mathfrak{a}$). Then there exists a proper analytic morphism*

$$\pi : (\widetilde{\mathscr{S}}, \widetilde{E}) \to (\mathscr{S}, E)$$

whose restriction to any relatively compact open set $\mathscr{S}_0$ of $\mathscr{S}$ coincide with a finite sequence of compatible blowings-up, such that all singular points of the strict transform $\mathscr{L}^{st}$ are isolated and elementary.

[4]The word *reduction*, instead of *resolution*, stands for the fact that it is impossible to eliminate singularities of a foliation via blowings-up. We can only hope to simplify their expressions.

Furthermore, given an irreducible component F of $\widetilde{E}$, either the line foliation $\mathscr{L}^{st}$ is everywhere transverse to F (and F is a dicritical component of $\mathscr{L}^{st}$) or $\mathscr{L}^{st}$ is everywhere tangent to F (and F is a non-dicritical component of $\mathscr{L}^{st}$).

Reduction of singularities of foliations in higher dimensions: Reduction of singularities of line foliation over *three-folds* is impossible (at least according to the above definition), as illustrated by an example of Sanz [Pan06, § 1.4]. By contrast, it is either possible to obtain a reduction of singularities via *weighted* blowings-up, as proved by Panazzolo and McQuillan in [Pan06, McQP13], or to reduce the singularities of the vector-field to a final list of non-elementary singularities, as proved by Cano, Roche and Spivakovsky in [CRS15]. The problem of reduction of singularities of vector-fields beyond an ambient variety of dimension three is open, but there are general results under extra hypothesis, e.g. Camacho, Cano and Sad's reduction of vector-fields with absolutely isolated singularities [CCS89].

Reduction of singularities of codimension one foliations over three-folds has been proved by Cano in [Can04], following a previous joint work with Cerveau [CaCe92] (where only non-dicritical foliations were considered). In higher dimensions the problem is open, but recent progress has been made by Cano and Duque in [CD18] where, roughly, the authors show that codimension one foliations admit a reduction of singularities along valuations of rank 1. This is a partial result towards the existence of a *local reduction of singularities.*

I am not aware of any general work about reduction of singularities of foliations of intermediate codimension (that is, neither codimenion one, nor dimension one) apart from works under strong hypothesis, e.g. my work [BdS18] about reduction of foliations which are totally integrable.

3.6 Quasianalytic classes

We mix the presentation given in [BdSBC18, Section 2] and [BdSB19, Section 3]. We consider a class of functions $\mathcal{Q}$ given by the association, to every open subset $U \subset \mathbb{R}^n$, of a subalgebra $\mathcal{Q}(U)$ of $\mathcal{C}^\infty(U)$ containing the restrictions to U of polynomial functions on $\mathbb{R}^n$, and closed under composition with a $\mathcal{Q}$-mapping (i.e., a mapping whose components belong to $\mathcal{Q}$). We assume that $\mathcal{Q}$ determines a sheaf of local $\mathbb{R}$-algebras of $\mathcal{C}^\infty$ functions on $\mathbb{R}^n$, for each n, which we also denote $\mathcal{Q}$.

Definition 3.6.1 (Quasianalytic classes)**.** *We say that $\mathcal{Q}$ is* quasianalytic *if it satisfies the following three axioms:*

1. Closure under division by a coordinate. *If $f \in \mathcal{Q}(U)$ and*

$$f(x_1, \ldots, x_{i-1}, a, x_{i+1}, \ldots, x_n) = 0,$$

   where $a \in \mathbb{R}$, then $f(x) = (x_i - a)h(x)$, where $h \in \mathcal{Q}(U)$.

2. Closure under inverse. *Let $\varphi : U \to V$ denote a $\mathcal{Q}$-mapping between open subsets U, V of $\mathbb{R}^n$. Let $a \in U$ and suppose that the Jacobian matrix $(\partial\varphi/\partial x)(a)$ is invertible. Then there are neighbourhoods U' of a and V' of $b := \varphi(a)$, and a $\mathcal{Q}$-mapping $\psi : V' \to U'$ such that $\psi(b) = a$ and $\psi \circ \varphi$ is the identity mapping of U'.*

3. Quasianalyticity. *If $f \in \mathcal{Q}(U)$ has Taylor expansion zero at $a \in U$, then f is identically zero near a.*

Remark 3.6.2.

1. *Axiom 3.6.1(1) implies that, if $f \in \mathcal{Q}(U)$, then all partial derivatives of f belong to $\mathcal{Q}(U)$.*

2. *Axiom 3.6.1(2) is equivalent to the property that the implicit function theorem holds for functions of class $\mathcal{Q}$. It implies that the reciprocal of a nonvanishing function of class $\mathcal{Q}$ is also of class $\mathcal{Q}$.*

The elements of a quasianalytic class $\mathcal{Q}$ will be called *quasianalytic functions*. A category of manifolds and mappings of class $\mathcal{Q}$ can be defined in a standard way. The category of $\mathcal{Q}$-manifolds is closed under blowing up whose centre is a $\mathcal{Q}$-submanifold [BM04]. As stated before, we can define a sheaf of quasianalytic functions, which we keep denoting by $\mathcal{Q}$. Given a point $\mathfrak{a} \in M$, we denote by $\mathcal{Q}_{\mathfrak{a}}$ the localization of $\mathcal{Q}$ to $\mathfrak{a}$, which is a local ring.

The axiomatic definition of quasianalytic classes is very recent [BM04, RSW03]. It captures the minimal conditions for a class of function to admit *resolution of singularities* [BM97, BM04] (that is, for Theorem 3.2.3 to hold in the quasianalytic category). We note that resolution of singularities of an ideal sheaf does not require the ideal sheaf be coherent as stated in Theorem 3.2.3, but only that the ideal sheaf is *privileged* (see [BMV15, Thm. 3.1], [BdSB19]), that is:

Definition 3.6.3 (Privileged ideal sheaf). *We say that a sheaf of ideals $\mathcal{I} \subset \mathcal{Q}$ (or, more generally, a sheaf of $\mathcal{Q}$-modules $\mathcal{M}$) is* privileged *if the stalk of $\mathcal{I}$ (or $\mathcal{M}$) at any point $\mathfrak{a}$ is generated by (possibly infinite) local sections defined in a common neighborhood of $\mathfrak{a}$.*

Note that if $\mathcal{I} \subset \mathcal{Q}$ is a privileged ideal sheaf, then $\mathcal{Q}/\mathcal{I}$ is quasicoherent.

The axiomatic definition also encompasses several classical classes of function which were studied in real-analysis, partial differential equations and model theory. There are two general examples of quasianalytic classes $\mathcal{Q}$ that are of particular interest to us:

I. Quasianalytic Denjoy-Carleman classes We use standard multiindex notation: Let $\mathbb{N}$ denote the nonnegative integers. If $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, we write $|\alpha| := \alpha_1 + \cdots + \alpha_n$, $\alpha! := \alpha_1! \cdots \alpha_n!$, $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, and $\partial^{|\alpha|}/\partial x^\alpha := \partial^{\alpha_1+\cdots+\alpha_n}/\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}$. We write (i) for the multiindex with 1 in the ith place and 0 elsewhere.

Definition 3.6.4. *Let $M = (M_k)_{k\in\mathbb{N}}$ denote a sequence of positive real numbers which is* logarithmically convex*; i.e., the sequence (M_{k+1}/M_k) is nondecreasing. A* Denjoy-Carleman class *$\mathcal{Q} = \mathcal{C}_M$ is a class of $\mathcal{C}^\infty$ functions determined by the following condition: A function $f \in \mathcal{C}^\infty(U)$ (where U is open in $\mathbb{R}^n$) is of class $\mathcal{C}_M$ if, for every compact subset K of U, there exist constants A, $B > 0$ such that*

$$\left|\frac{\partial^{|\alpha|} f}{\partial x^\alpha}(\mathfrak{a})\right| \leq AB^{|\alpha|}\alpha! M_{|\alpha|}, \qquad \text{for all } \mathfrak{a} \in K \text{ and } \alpha \in \mathbb{N}^n. \tag{3.1}$$

We use the notation M to denote a sequence (as opposed to a manifold or smooth space) only in this section, in order to be consistent with standard notation for Denjoy-Carleman classes.

Remark 3.6.5. *The logarithmic convexity assumption implies that $M_j M_k \leq M_0 M_{j+k}$, for all j, k, and that the sequence $((M_k/M_0)^{1/k})$ is nondecreasing. The first of these conditions guarantees that $\mathcal{C}_M(U)$ is a ring, and the second that $\mathcal{C}_M(U)$ contains the ring $\mathcal{O}(U)$ of real-analytic functions on U, for every open $U \subset \mathbb{R}^n$. (If $M_k = 1$, for all k, then $\mathcal{C}_M = \mathcal{O}$.)*

If X is a closed subset of U, then $\mathcal{C}_M(X)$ will denote the ring of restrictions to X of $\mathcal{C}^\infty$ functions which satisfy estimates of the form (3.1), for every compact $K \subset X$.

A Denjoy-Carleman class $\mathcal{Q} = \mathcal{C}_M$ is a quasianalytic class in the sense of Definition 3.6.1 if and only if the sequence $M = (M_k)_{k\in\mathbb{N}}$ satisfies the following two assumptions in addition to those of Definition 3.6.4.

(a) $\sup \left(\dfrac{M_{k+1}}{M_k}\right)^{1/k} < \infty$.

(b) $\displaystyle\sum_{k=0}^{\infty} \frac{M_k}{(k+1)M_{k+1}} = \infty$.

Assumption (a) implies that $\mathcal{C}_M$ is closed under differentiation. The converse of this statement is due to S. Mandelbrojt [Man52]. In a Denjoy-Carleman class $\mathcal{C}_M$, closure under differentiation is equivalent to the axiom 3.6.1(1) of closure under division by a coordinate—the converse of Remark 3.6.2(1) is a consequence of the fundamental theorem of calculus.

According to the Denjoy-Carleman theorem, the class $\mathcal{C}_M$ is quasianalytic (axiom 3.6.1(3)) if and only if the assumption (b) holds (see, e.g. [Hor83, Thm. 1.3.8]).

The closure of a Denjoy-Carleman class $\mathcal{C}_M$ under composition is due to Roumieu [Rou62] and the closure under inverse to Komatsu [Kom79]; see [BM04] for simple proofs. A Denjoy-Carleman class $\mathcal{Q} = \mathcal{C}_M$ satisfying the assumptions (a) and (b) above is thus a quasianalytic class, in the sense of Definition 3.6.1.

The following criteria are due to Cartan and Mandelbrojt (see e.g. [Man52, Thm. XI]). If $\mathcal{C}_M$, $\mathcal{C}_N$ are Denjoy-Carleman classes, then $\mathcal{C}_M(U) \subseteq \mathcal{C}_N(U)$, for all U, if and only if $\sup (M_k/N_k)^{1/k} < \infty$; in this case, we write $\mathcal{C}_M \subseteq \mathcal{C}_N$. For any given Denjoy-Carleman class $\mathcal{C}_M$, there is a function in $\mathcal{C}_M((0,1))$ whose germ at any point $\mathfrak{a} \in (0,1)$ is not in any given smaller class [J16, Thm. 1.1].

Given $M = (M_j)_{j\in\mathbb{N}}$ and a positive integer p, let $M^{(p)}$ denote the sequence $M_j^{(p)} := M_{pj}$. If M is logarithmically convex, then $M^{(p)}$ is logarithmically convex:

$$\frac{M_{kp}}{M_{(k-1)p}} = \frac{M_{kp}}{M_{kp-1}} \cdots \frac{M_{kp-p+1}}{M_{kp-p}} \leq \frac{M_{kp+p}}{M_{kp+p-1}} \cdots \frac{M_{kp+1}}{M_{kp}} = \frac{M_{(k+1)p}}{M_{kp}}.$$

Therefore, if $\mathcal{C}_M$ is a Denjoy-Carleman class, then so is $\mathcal{C}_{M^{(p)}}$. Clearly, $\mathcal{C}_M \subsetneqq \mathcal{C}_{M^{(p)}}$. Moreover, the assumption (a) above for $\mathcal{C}_M$ immediately implies the same condition for $\mathcal{C}_{M^{(p)}}$. In general, however, it is not true that assumption (b) (i.e., the quasianalyticity axiom (3)) for $\mathcal{C}_M$ implies (b) for $\mathcal{C}_{M^{(p)}}$ [Now15, Example 6.6]. In particular, in general, $\mathcal{C}_{M^{(p)}} \supsetneq \mathcal{C}_M$. Moreover, $\mathcal{C}_{M^{(2)}}$ is the smallest Denjoy-Carleman class containing all $g \in \mathcal{C}^\infty(\mathbb{R})$ such that $g(t^2) \in \mathcal{C}_M(\mathbb{R})$ [Now15, Rmk. 6.2].

II. Model theory We give a geometrical presentation, which specialize certain definitions of logic to real-sets. A collection of sets $\mathcal{S} = (\mathcal{S}_n)$, where each $\mathcal{S}_n$ is a collection of sub-sets of $\mathbb{R}^n$, is said to be a *structure* if these collections are closed under the usual boolean operations (union, intersection, complement) and linear projections parallel to a coordinate axis. A structure is said to be *o-minimal* if all sets of $\mathcal{S}$ have at most a finite number of connected components. A function $g : U \subset \mathbb{R}^n \to \mathbb{R}^m$ is said to be *definable* in a structure $\mathcal{S}$ if its graph $\Gamma(f)$ is a set in $\mathcal{S}$ (in particular, note that U must be a definable set). Finally, a structure is said to be polynomially bounded, if all continuous definable functions $f : \mathbb{R} \to \mathbb{R}$ have at most polynomial growth at infinity, that is, there exists $p \in \mathbb{N}$ such that $\lim_{x\to\infty} f(x)/x^p = 0$.

At the one hand, classes of C^∞-functions locally definable in a given polynomially-bounded o-minimal structure $\mathcal{S}$ define quasianalytic classes $\mathcal{Q}$, according to a result of C. Miller [Mi95]. On the other hand, the structure generated by restricted functions[5] in a given quasianalytic class $\mathcal{Q}$, which we denote by $\mathbb{R}_{\mathcal{Q}}$, is *o-minimal* and *polynomially bounded*, according to a result of Rolin, Speissegger and Wilkie [RSW03]. Recently, Rolin and Servi have provided the necessary axiomatic framework for algebras of (not necessarily C^∞) functions to generate an o-minimal structure [RS15].

We finish this section by the following result about o-minimal structures generated by quasianalytic Denjoy-Carleman classes, which provides a connection between the two examples:

Theorem 3.6.6 (Belotto da Silva, Bierstone, Chow [BdSBC18])**.** *Let $\mathcal{C}_M$ denote a quasianalytic Denjoy-Carleman class. If $f \in \mathcal{C}^\infty(W)$, where W is open in $\mathbb{R}^n$, and f is definable in $\mathbb{R}_{\mathcal{C}_M}$, then there exists $p \in \mathbb{N}$ such that $f \in \mathcal{C}_{M^{(p)}}(W)$.*

3.7 Properties of quasianalytic classes

Somehow surprisingly, several algebraic properties of analytic functions do not extend, or are not known to extend, to general quasianalytic classes. We highlight three of these properties:

(Failure of the) Weierstrass preparation property We recall that a class of functions $\mathcal{Q}$ is said to satisfy the *Weierstrass preparation* property

[5]More precisely, given a function $f : U \subset \mathbb{R}^n \to \mathbb{R} \in \mathcal{Q}(U)$ where $[0,1]^n \subset U$, we consider its restriction to $[0,1]^n$, that is, $\widetilde{f}(x) = f(x)$ if $x \in [0,1]^n$ and $\widetilde{f}(x) = 0$ otherwise.

if, for every function $f \in \mathcal{Q}(V)$ (where $V \subset \mathbb{R}^n$) and point $\mathfrak{a} \in V$ such that there exists $d \in \mathbb{N}$ so that $\partial_{x_1}^d(f)(\mathfrak{a}) \neq 0$, there exists a neighbourhood W of $\mathfrak{a}$ and functions ξ and $f_i \in \mathcal{Q}(W)$ such that:

$$f(x) = \xi(x)\left[x_1^d + \sum_{i=0}^{d-1} f_i(x_2, \ldots, x_n) \cdot x_1^i\right]$$

where $\xi(\mathfrak{b}) \neq 0$ for all $\mathfrak{b} \in V$. Analytic functions do satisfy this property by the Weierstrass preparation Theorem (and C^∞-functions in general by the Malgrange Preparation Theorem). It turns out that Weierstrass property does not hold in general quasianalytic classes [PR13]. For Denjoy-Carleman classes, moreover, it is not possible to achieve the Weierstrass preparation property even with *loss of regularity*, that is, even if we allow the functions ξ and f_i to belong to a quasianalytic class $\mathcal{Q}'$ which properly contains the original class $\mathcal{Q}$ [NSV04, ABBNZ14].

(Non-)extendability property The Weierstrass preparation property fails for general quasianalytic classes because, essentially, these classes admit no *extension property*. More precisely, Nazarov, Sodin and Volberg [NSV04, § 5.3] have constructed a Denjoy-Carleman class $\mathcal{C}_M$ and a function $g \in \mathcal{C}_M([0,1))$ which admits no quasianalytic extension to a neighbourhood of 0, that is, for every $\delta > 0$, every quasianalytic Denjoy-Carleman class $\mathcal{C}_{M'} \supset \mathcal{C}_M$, and every function $\widetilde{g} \in \mathcal{C}_{M'}((-\delta, 1))$, we have that $\widetilde{g}|_{[0,1)} \neq g$. No such example exists for analytic function, where the convergence of the Taylor expansion always guarantees extension to a neighborhood.

The lack of extension can be exploited, in a simple way, to construct a counter-example to the Weierstrass preparation property with arbitrary loss of regularity (see, e.g. [BdSBB17, Remark 7.5(3)]). In particular, the above example of Nazarov, Sodin and Volberg shows that there *exists a quasianalytic Denjoy-Carleman class* $\mathcal{C}_M$ which does not admit the Weierstrass preparation property, even with loss of regularity. In [ABBNZ14, 2014], the authors have showed that the lack of extension holds in arbitrary quasianalytic Denjoy-Carleman classes via elementary methods, even with a prescribed loss of regularity. In particular, they conclude that the Weierstrass preparation property fails for *all quasianalytic Denjoy-Carleman classes* $\mathcal{C}_M$. Finally, Rolin and Parusiński in [PR13, 2016] constructed a function $g \in \mathcal{Q}([0,1))$ which admits no quasianalytic extension to $\mathcal{Q}((-\delta,1))$ for an arbitrary quasianalytic class $\mathcal{Q}$ via elementary complex-analysis arguments, effectively showing that the Weierstrass preparation property fails for *all quasianalytic classes* $\mathcal{Q}$ which properly contain the analytic functions.

Noetherianity of local rings Recall that a commutative ring R is said to be Noetherian if every ideal $I \subset R$ admits a finite set of generators. Roughly, noetherianity provides a notion of "algebraic compactness", which is a desirable technical property in algebraic geometry. Noetherianity of local rings in the algebraic category follows from the Hilbert basis Theorem, and in the analytic category, all proofs (that I am aware of) use the Weierstrass preparation property to reduce to the polynomial case, and conclude via an argument "à la Hilbert basis". It follows that all proofs of Noetherianity of local analytic rings (that I am aware of) do not extend, in a trivial way, to general quasianalytic classes.

In dimension one, that is, if we consider a quasianalytic ring $\mathcal{Q}(U)$ where $U \subset \mathbb{R}$, local quasianalytic rings $\mathcal{Q}_{\mathfrak{a}}$ are Noetherian by axiom 3.6.1(1). In higher dimensions, nevertheless, Noetherianity is not known to hold and the problem goes back to, at least, [Chi76]. At the current date, the reader may find at least two pre-prints in arXiv claiming that such local rings are not necessarily Noetherian, but (as far as I am aware) both papers have gaps which can not be fixed in a simple way. The problem seems to be particularly difficult because it is intrinsic to higher dimensions, while most non-trivial constructions of quasianalytic functions are one-dimensional.

Chapter 4

Resolution of singularities and log-differentials

4.1 Introduction

Since at least the beginning of the XXth century, mathematicians have been interested in extending resolution of singularities to various settings related to differential equations [Be1901]. For instance, resolution of singularities of foliations is the basic ingredient in several classical and modern problems in dynamical systems and algebraic geometry (e.g. the Hilbert 16^{th} problem [Du1923], [DR91] [Rou95] and the minimal model program for foliations [McQ08]). A more subtle motivation arises when the differential equations have an *indirect* role. For instance, Giraud argues that a proof of resolution of singularities in positive characteristics would demand a considerable control of the differential forms [Gi83]; and there is an accumulation of evidence that controlling the *derivatives tangent to morphism* [BdSB19] (a notion that coincides with that of *relative log derivations* of origin in work of Grothendieck and Deligne, see [O18, Ch. IV], and used in [ATW20a], [ATW20b]) is crucial when dealing with monomialization of morphisms and resolution of singularities in families [BdSB19], [ATW20a], [ATW20b].

Unfortunately, differential equations do not transform well by blowings-up, and most results in resolution of singularities of differential equations are either *local*, or restricted to *low dimensions*, up to three. Indeed, consider a coherent sub-sheaf Δ of derivations (or vector-fields). In contrast to ideal sheaves, the transform of different elements of Δ by blowings-up are *asymmetric*. For example, let us consider in $\mathbb{C}^3$, with coordinate system (x_1, x_2, x_3), the sub-sheaf of derivations Δ generated by two regular vector fields $(\partial_{x_2}, \partial_{x_3})$. Let $\sigma : M \to \mathbb{C}^3$ be the blowing-up with center $(x_1 = x_2 = 0)$, and consider the x_1-chart (which is given by $x_1 = y_1$, $x_2 = y_1 \cdot y_2$ and $x_3 = y_3$) where the pull-back of Δ is given by:

$$\sigma^*(\Delta) = \left(\frac{1}{y_1}\partial_{y_2}, \partial_{y_3} \right).$$

Note that there is an asymmetry between the transform of the generators: there is a pole in the pull-back of ∂_{x_2}, but not in the pull-back of ∂_{x_3} (see example 4.6.7.1 below for an extended discussion).

Because of this asymmetry, it is difficult to adapt, at least in a direct way, the approach of Hironaka to derivations. The lack in comprehension of this phenomenon is behind some early missteps (see some remarks in this direction in [BdSBGM17, page 785]). Recent progress in reduction of singularities of foliations are based either in weighted blowings-up [Pan06, McQP13] (where the weights can be used to "compensate" the asymmetry), or in valuation techniques [CRS15, CD18] (which control the asymmetry via combinatorial methods), and the situation is similar for the monomialization of morphisms [Cu99, Cu02, Cu05, Cu07, Cu15, Cu17, BdSB19, ATW20a, ATW20b].

In the past seven years, I developed techniques of *resolution of singularities adapted to log-differentials*, that is, methods of resolution of singularities which take in consideration the asymmetric behavior of a sub-sheaf of log-derivations under blowings-up, see §4.6. We used these methods to study: resolution of singularities of the cotangent sheaf [BdSBGM17] [BdSB17], in collaboration with Bierstone, Grandjean and Milman, see §4.2; local monomialization of Darboux-type first integrals [BdS18], see §4.3; and monomialization of quasianalytic morphisms [BdSB19], in collaboration with Bierstone, see §4.4. I finish the chapter by making an overview on two overarching methods: *logarithmic Fitting ideals*, see §4.5, and of techniques of *resolution of singularities adapted to log-differentials*, see §4.6.

4.2 Resolution of singularities of the cotangent sheaf [BdSBGM17, BdSB17]

Let X_0 denote a complex- or real-analytic coherent space (or an algebraic variety over a field of characteristic zero), and assume that X_0 is reduced. We consider the following Conjecture, whose formulation is due to Youssin [Y98]:

Conjecture 4.2.1 (*Resolution of singularities of the cotangent sheaf* [Y98]). *There is a resolution of singularities of X_0:*

$$\sigma : (X, E) \to (X_0, Sing(X_0))$$

such that the pulled-back of the cotangent sheaf $\Omega^1_{X_0}$ (that is, the sub-sheaf of Ω^1_X generated by $\sigma^*(\Omega^1_{X_0}) \otimes \mathcal{O}_X$) is locally generated by differential monomials*:*

$$d(\boldsymbol{u}^{\boldsymbol{\alpha}_i}),\ i = 1, \dots, s,\ \text{ and }\ d(\boldsymbol{u}^{\boldsymbol{\beta}_j} v_j),\ j = 1, \dots, n-s, \tag{4.1}$$

where $n = \dim X_0$, $(\boldsymbol{u}, \boldsymbol{v}) = (u_1, \dots, u_s, v_1, \dots, v_{n-s})$ are local analytic (or étale) coordinates on X, which we call Hsiang-Pati *coordinates, and*

1. $\operatorname{supp} E = (u_1 \cdots u_s = 0)$,
2. *the multiindices $\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_s \in \mathbb{N}^s$ are linearly independent over $\mathbb{Q}$,*
3. *$\{\boldsymbol{\alpha}_i, \boldsymbol{\beta}_j\}$ is totally ordered (with respect to the componentwise partial ordering of $\mathbb{N}^s$).*

The above Conjecture was previously proved (at least locally) in the case of surfaces with isolated singularities by W.-C. Hsiang and V. Pati in [HP85] and a more conceptual proof in this case was given by W. Pardon and M. Stern in [PS01]. Nowadays, the problem is sometimes called the "Hsiang-Pati problem". In our work [BdSBGM17], we prove the Conjecture for three-folds:

Theorem 4.2.2 (Belotto da Silva, Bierstone, Grandjean and Milman). *Conjecture 4.2.1 holds for varieties X_0 of dimension ≤ 3.*

The proof of Theorem 4.2.2 relies on the techniques of logarithmic Fitting ideals and resolution of singularities adapted to log-differentials, presented in § 4.5 and 4.6 respectively. In particular, we provide an intrinsic characterization of the existence of Hsiang-Pati coordinate systems in term of logarithmic Fitting ideals, in any dimension, in Theorem 4.5.3 below. We postpone its discussion to §4.5, where we define logarithmic Fitting ideals.

One of the main interests of Conjecture 4.2.1 has been for applications to the L^2-cohomology of the smooth part of a singular variety, going back to the original ideas of Cheeger [Che79, Che80]. Hsiang and Pati used their result to prove that the intersection cohomology (with the middle perversity) of a complex surface X_0 equals the L^2-cohomology of $X_0 \setminus \operatorname{Sing}(X_0)$ (Cheeger-Goresky-MacPherson conjecture [CGM82]). A positive answer to the Conjecture would have important consequences to the study of the L^2-cohomology of X_0 and, more generally, to metric properties of X_0. Indeed, the Conjecture implies a reduction of singularities (up to local bi-Lipschitz equivalence) of the inner metric over X_0:

Reduction of singularities of the inner metric: Suppose that X_0 is a complex embedded sub-variety of a smooth complex-analytic manifold M endowed with a (smooth) Hermitian metric ds^2. We recall that, in local coordinate systems $\boldsymbol{z} = (z_1, \dots, z_N)$ centered at a point $\mathfrak{a} \in M$, we can write:

$$ds^2 = \sum_{i,j=1}^{N} h_{ij}(\boldsymbol{z}) dz_i \otimes d\overline{z_j}$$

where the matrix $H(\boldsymbol{z}) = [h_{ij}(\boldsymbol{z})]$ is everywhere a positive-defined Hermitian matrix. The *inner metric* over X_0 induced by ds^2, which we denote by $d^{in}_{X_0}$, is defined as follows:

$$d^{in}_X(\mathfrak{a}, \mathfrak{b}) = \inf\{\text{length}_{ds^2}(\gamma); \gamma : [0,1] \to X \text{ cont. of BV}, \gamma(0) = \mathfrak{a}, \gamma(1) = \mathfrak{b}\}$$

(where "cont. of BV" stands for "continuous of bounded variation"). Note that $(X_0, d^{in}_{X_0})$ is a metric space. Now, given an homeomorphism $\Phi : X_0 \to Y_0$ between complex varieties endowed with metrics d_{X_0} and d_{Y_0} respectively, we say that Φ is a *bi-Lipschitz* homeomorphism if there exists a constant $K \in \mathbb{R}^+$ such that for every $\mathfrak{a}$, $\mathfrak{b} \in X_0$:

$$\frac{1}{K} \cdot d_{Y_0}(\Phi(a), \Phi(b)) \leq d_{X_0}(a, b) \leq K \cdot d_{Y_0}(\Phi(a), \Phi(b))$$

In this case, we say that (X_0, d_{X_0}) and (Y_0, d_{Y_0}) are bi-Lipschitz *equivalent*.

The validity of Conjecture 4.2.1 would imply existence of resolution of singularities $(X, E) \to (X_0, \mathrm{Sing}(X_0))$ such that, at every point $\mathfrak{a} \in X$ there exists a coordinate system $(\boldsymbol{u}, \boldsymbol{v})$ (the Hsiang-Pati coordinates) defined in an open neighborhood $U_{\mathfrak{a}}$ of $\mathfrak{a}$ such that the pull-back of the ambient Hermitian metric $\sigma^*(ds^2)$ (and, therefore, of the inner metric) restricted to $U_{\mathfrak{a}}$ is bi-Liptchitz equivalent to:

$$\sum_{i=1}^{s} d(\boldsymbol{u}^{\boldsymbol{\alpha}_i}) \otimes \overline{d(\boldsymbol{u}^{\boldsymbol{\alpha}_i})} + \sum_{j=1}^{n-s} d(\boldsymbol{u}^{\boldsymbol{\beta}_j} v_j) \otimes \overline{d(\boldsymbol{u}^{\boldsymbol{\beta}_j} v_j)}. \tag{4.2}$$

This normal form plays a crucial role in the study of W.-C. Hsiang and V. Pati [HP85, 1985] about the L^2-cohomology of X_0, and their proof of the Cheeger-Goresky-MacPherson Conjecture. Several controversial articles on both the Hsiang-Pati problem and the L_2-cohomology of singular varieties have perhaps discouraged work on these questions; we hope that our result, Theorem 4.2.2, will lead to a renewal of interest on this Conjecture, which is open for varieties of dimension bigger or equal to three.

We note, furthermore, that the above normal form plays an important role in our work in collaboration with Figalli, Parusiński and Rifford about the Sard Conjecture in sub-Riemannian geometry [BdSFPR18], see §6 and Theorem 6.9.1.

4.3 Monomialization of Darboux type first integrals [BdS18]

Let $\mathcal{K}$ be a sub-field of $\mathbb{C}$ and M be a coherent and smooth complex- or real-analytic space. We say that a singular foliation $\mathcal{F}$ admits l first integrals of $\mathcal{K}$-Darboux type over a point $\mathfrak{a} \in M$ if the foliation $\mathcal{F}$ is tangent to

$$\omega_i := \sum_{j=1}^{t} k_{ij} \frac{dg_{ij}}{g_{ij}}, \quad k_{ij} \in \mathcal{K}, \text{ and } g_{ij} \in \mathcal{O}_{\mathfrak{a}}, i = 1, \ldots, l$$

for some $t \in \mathbb{N}$, where $\omega_1 \wedge \ldots \wedge \omega_l \not\equiv 0$. Equivalently, there exists l complex multi-valued function germs $f_i := \prod_{j=1}^{t} g_{ij}^{k_{ij}}$ such that $df_1 \wedge \ldots \wedge df_l \not\equiv 0$ and $\partial(f_i) \equiv 0$ for every differential ∂ tangent to $\mathcal{F}$.

Remark 4.3.1. *A "generalized" Darboux first integral, in this work, denotes a first integral of the form*

$$f = exp\,(\phi/\psi) \prod g_j^{k_j} \qquad \text{where } \phi,\ \psi \text{ are analytic germs.}$$

In some references, e.g. [Nov09], the notion of Darboux first integral also includes, what we call, "generalized" Darboux first integrals.

As an application of resolution of singularities adapted to log-differentials (see §4.6), we prove the existence of a local monomialization of Darboux first integrals. More precisely:

Theorem 4.3.2 (Belotto da Silva [BdS18])**.** *Let M be a smooth complex- or real-analytic space, D be a SNC divisor over M, and $\mathcal{F}$ be a foliation over M. Fix $\mathfrak{a} \in M$, and suppose that the singular foliation $\mathcal{F}$ admits l-first integrals of $\mathcal{K}$-Darboux type. There exists a neighborhood V of $\mathfrak{a}$ and a finite collection of morphisms $\sigma_\lambda : (V_\lambda, D_\lambda) \to (V, D)$ such that:*

1. *each σ_λ is a composite of finitely many smooth local blowings-up compatible with D;*

2. *the family of morphisms $\{\sigma_\lambda\}$ cover V and there are compact subsets $K_\lambda \subset V_\lambda$ such that $\bigcup \sigma_\lambda(K_\lambda)$ is a compact neighbourhood of $\mathfrak{a}$;*

3. *The strict transform $\mathcal{F}_\lambda$ of the singular foliation $\mathcal{F}$ admits l monomial first integrals with exponents in $\mathcal{K}$, that is, at every point $\mathfrak{b} \in V_\lambda$, there exists a coordinate system $(\boldsymbol{u}, \boldsymbol{v}) = (u_1, \ldots, u_r, v_1, \ldots, v_s)$, with $r + s = n$, such that $D_\lambda = (u_1 \cdots u_r = 0)$ and*

$$f_i = \boldsymbol{u}^{\alpha_i} = u_1^{\alpha_{i,1}} \cdots u_r^{\alpha_{r,1}}, \qquad \text{where } \alpha_i \in \mathcal{K}^r \quad \text{for } i = 1, \ldots, l$$

are first integrals of $\mathcal{F}_\lambda$ *such that* $d\boldsymbol{u}^{\alpha_1} \wedge \ldots \wedge d\boldsymbol{u}^{\alpha_r} \not\equiv 0$. *If* $\mathcal{K} = \mathbb{Q}$, *furthermore, then* $\alpha_i \in \mathbb{N}^r$ *for* $i = 1, \ldots, l$.

This is one of the few results about reduction of foliations without leaf-dimensions hypothesis (but under the strong hypothesis of integrability). One of the original motivations of this result was the study of pseudo-abelian integrals, following Braghta's thesis [Br13] and personal communication from P. Mardesic. Pseudo-abelian integrals are important technique in order to estimate the number of limit cycles which bifurcate from a Darboux planar vector-field, c.f. the works of Bobieński, Mardesic and Novikov [BoMa08, Nov09]. Note that if the first integrals are algebraic or analytic, then the above Theorem can be obtained as a consequence of monomialization of morphisms [Cu99, Cu05, Cu07, BdSB19], c.f. §4.4.

4.4 Monomialization of quasianalytic morphisms [BdSB19]

In collaboration with E. Bierstone, we prove that a mapping in a real quasianalytic class (see 3.6.1) can be transformed by sequences of simple changes of the source and target (sequences of local blowings-up and local power substitutions) to a mapping whose components are monomials with respect to suitable local coordinate systems. In the general algebraic and analytic cases, we show, moreover, that only local blowings-up are needed for monomialization. The proof of our results demand us to improve the techniques of quasianalytic continuation (see §5.6), logarithmic Fitting Ideals (see §4.5) and resolution of singularities adapted to log-differentials (see §4.6).]

In what follows, we denote by $\mathcal{Q}$ either a quasianalytic category, or the complex- or real-analytic category, or the algebraic category over a field of characteristic zero. One of the advantages of working with quasianalytic classes is that the axiomatic framework given in Definition 3.6.1 includes all of the above categories. In the algebraic case, extra care has to be taken when dealing with coordinate systems, which are usually étale and not regular. We refer the reader to [BdSB19, §3.1] for the necessary adaptations, and we focus the presentation on the real-quasianalytic and/or analytic classes.

Let M, N denote manifolds of class $\mathcal{Q}$; say $m = \dim M$, $n = \dim N$; and D and E be SNC (reduced) divisors on M and N respectively. A $\mathcal{Q}$-*morphism* $\Phi : (M, D) \to (N, E)$ denotes a $\mathcal{Q}$-mapping $\Phi : M \to N$ such that $\Phi^{-1}(E)$ is SNC *as a space* and its zero locus lies in that of D (i.e., the ideal of $\Phi^{-1}(E)$ is principal monomial). We start by defining the notion of monomial morphisms:

Definition 4.4.1 (Monomial morphism)**.** *We say that a $\mathcal{Q}$-morphism $\Phi : (M, D) \to (N, E)$ is **monomial** at a point $\mathfrak{a} \in M$ if there are $\mathcal{Q}$-coordinate systems, which we call* Φ-monomial,

$$\begin{aligned} (\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) &= (u_1, \ldots, u_r, v_1, \ldots, v_s, w_1, \ldots, w_t), \\ (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) &= (x_1, \ldots, x_p, y_1, \ldots, y_q, z_{q+1}, \ldots, z_{s'}), \end{aligned} \tag{4.3}$$

centred at $\mathfrak{a}$ and $\mathfrak{b} = \Phi(\mathfrak{a})$ (respectively), where $r + s + t = m$, $p + s' = n$ and $q \leq s \leq s'$, adapted to D and E (that is, $D = \{u_1 \cdots u_r = 0\}$ and $E = \{x_1 \cdots x_p \cdot y_1 \cdots y_q = 0\}$) in which Φ can be written

$$\begin{aligned} x_j &= \boldsymbol{u}^{\boldsymbol{\alpha}_j}, && j = 1, \ldots, p, \\ y_k &= \boldsymbol{u}^{\boldsymbol{\beta}_k}(\xi_k + v_k), && k = 1, \ldots, q, \\ z_l &= v_l, && l = q + 1, \ldots, s, \\ z_l &= 0, && l = s + 1, \ldots, s', \end{aligned} \tag{4.4}$$

where

1. *the exponent vectors $\boldsymbol{\alpha}_j = (\alpha_{j1}, \ldots, \alpha_{jr}) \in \mathbb{N}^r$, $j = 1, \ldots, p$, are linearly independent over $\mathbb{Q}$ (in particular, $r \geq p$);*
2. *for each $k = 1, \ldots, q$, $\boldsymbol{\beta}_k = (\beta_{k1}, \ldots, \beta_{kr}) \in \mathbb{N}^r$ is nonzero and $\mathbb{Q}$-linearly dependent on $\{\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_p\}$, and $\xi_k \neq 0$.*

We say that Φ is a monomial morphism *if it is monomial at every point $\mathfrak{a} \in M$.*

Remark 4.4.2. *Given a monomial morphism $\Phi : (M, D) \to (N, E)$ and a point $\mathfrak{a} \in M$, there exists a neighborhood $U_{\mathfrak{a}}$ of $\mathfrak{a}$ such that the image $\Phi(U_{\mathfrak{a}})$ is contained on a (locally defined) smooth $\mathcal{Q}$-sub-variety Σ of N, of dimension equal to the generic rank of Φ, and which has normal crossings with D.*

Our objective is to study the following problem:

Problem 1 (Monomialization problem)**.** *Let $\Phi : (M, D) \to (N, E)$ be a proper $\mathcal{Q}$-morphism. Does there exist a commutative diagram*

$$\begin{array}{ccc} (\widetilde{M}, \widetilde{D}) & \xrightarrow{\ \sigma\ } & (M, D) \\ \big\downarrow{\scriptstyle \widetilde{\Phi}} & & \big\downarrow{\scriptstyle \Phi} \\ (\widetilde{N}, \widetilde{E}) & \xrightarrow{\ \tau\ } & (N, E) \end{array} \tag{4.5}$$

where σ and τ are composites of (global) blowings-up with smooth centres and the induced map $\widetilde{\Phi}$ is monomial?

The monomialization problem in the algebraic category has an extensive literature, with roots in the problem of factorization of birational morphims[1], and goes back at least to works of Abhyankar [A56], Zariski [Z58] and Hironaka [Hi64]. The literature includes the works on mappings of complex surfaces of Akbulut and King [AK92] and the local monomialization theorems of Cutkosky [Cu99, Cu05, Cu17] and Denef [Dene13]. In addition to his local results, Cutkosky has proved monomialization by global blowings-up for dominant projective morphisms in dimension three [Cu02, Cu07, Cu15], effectively given a positive answer to Problem 1 in the algebraic category when M and N have dimension at most three. We note that the proofs of Cutkosky do not adapt, in an evident way, to the analytic category because they rely in extension techniques, e.g. Bertini type Theorems. Variants of the monomialization problem, going back to the works of Kempf, Knudsen, Mumford and Saint-Donat [KKMS73], include the relative desingularization of morphisms via Kummer blowings-up of Abramovich, Temkin and Włodarczyk [ATW20a], [ATW20b] and the semi-stable reduction results of Abramovich, Adisprasito, Karu, Liu, and Temkin [AK00], [ALT20].

In collaboration with Bierstone, our goal was to study the monomialization problem 1 in the general quasianalytic setting. A positive answer would not only have consequences on classical problems in algebraic geometry, such as factorization of birational morphisms, but also applications to o-minimal geometry and model theory (some of them are discussed in §5.5 below). Our first result provides a negative answer to problem 1 in its full generality:

Theorem 4.4.3 (Belotto da Silva, Bierstone [BdSB19])**.** *A proper morphism $\varphi : M \to N$ of real-analytic manifolds cannot, in general, be monomialized by global blowings-up of the source and target, even over some neighbourhood of a given point of $\varphi(M)$. More precisely, there exists a proper real-analytic morphism $\varphi : M \to N$ where $dim(M) = 3$ and $dim(N) = 4$ which can not be monomialized by global blowings-up with smooth centers.*

[1] Following the notation of Problem 1, we further assume that Φ is a birational dominant morphism, and we ask if we can find a commutative diagram (4.5) satisfying the stronger thesis that $\widetilde{\Phi}$ is the identity. See the work of Zariski [Z58] for the two dimensional case, and the works of Abramovich, Denef, Karu, Matsuki and Włodarczyk [W03], [ADK13], [AKMW02] for weak-factorization in arbitrary dimensions.

In fact, we show that φ cannot, in general, be transformed by global blowings-up of the source and target to a mapping that is *regular in the sense of Gabrielov*[2] whenever $\dim(M) \geq 3$ and $\dim(N) \geq 4$. Our counterexample can be found in [BdSB19, §2.1] and is inspired by an example of Bierstone and Parusiński [BP18]. We note that the same phenomenon does not happen in the complex-analytic and algebraic categories, where every proper morphism is regular in the sense of Gabrielov according to results of Remmert [Re57] (in the complex-analytic case) and Chevalley [Ch43] (in the algebraic case).

Next, we proved the existence of a local monomialization of morphisms in the generality of quasianalytic categories. More precisely:

Theorem 4.4.4 (Belotto da Silva, Bierstone [BdSB19])**.** *Suppose that $\mathcal{Q}$ is a **real quasianalytic** class. Let $\mathfrak{a} \in M$. Then there is an open neighbourhood V of $\mathfrak{a}$ and a finite number of commutative diagrams*

$$\begin{array}{ccc} (V_\lambda, D_\lambda) & \xrightarrow{\sigma_\lambda} & (M, D) \\ \big\downarrow{\scriptstyle \Phi_\lambda} & & \big\downarrow{\scriptstyle \Phi} \\ (W_\lambda, E_\lambda) & \xrightarrow{\tau_\lambda} & (N, E) \end{array} \tag{4.6}$$

where

1. *each σ_λ and τ_λ is a composite of finitely many smooth **local blowings-up** and **local power substitutions**, compatible with D and E, respectively;*

2. *the families of morphisms $\{\sigma_\lambda\}$ and $\{\tau_\lambda\}$ cover V and W, respectively, and there are compact subsets $K_\lambda \subset V_\lambda$, $L_\lambda \subset W_\lambda$, such that $\bigcup \sigma_\lambda(K_\lambda)$ and $\bigcup \sigma_\lambda(L_\lambda)$ are (compact) neighbourhoods of $\mathfrak{a}$ and $\mathfrak{b}$, respectively;*

3. *each Φ_λ is a monomial morphism (i.e., locally of the form* (4.4)*, where $D_\lambda = \{u_1 \cdots u_r = 0\}$ and $E_\lambda = \{x_1 \cdots x_p \cdot y_1 \cdots y_q = 0\}$).*

Theorem 4.4.5 (Belotto da Silva, Bierstone [BdSB19], Cutkosky [Cu99, Cu05, Cu17])**.** *Suppose that $\mathcal{Q}$ is either the class of $\mathbb{R}$**- or** $\mathbb{C}$**-analytic functions**, or the class of **algebraic functions over a field** $\mathbb{K}$ **of characteristic zero**. Let $\mathfrak{a} \in M$. Then there is an open neighbourhood V of $\mathfrak{a}$ and a finite number of commutative diagrams* (4.6)*, where*

[2] A morphism $\varphi : V \to W$ of real-analytic manifolds is *regular* at a point $\mathfrak{a} \in V$, in the sense of Gabrielov [G73], if $\dim \mathcal{O}_{\varphi(\mathfrak{a})} / \ker \varphi^*_{\mathfrak{a}}$ equals the generic rank of φ at $\mathfrak{a}$. We say that $\varphi : V \to W$ is *regular* if it is regular at every point of V.

1. *each σ_λ and τ_λ is a composite of finitely many smooth* ***local blowings-up****, compatible with D and E, respectively;*

2. *the families of morphisms $\{\sigma_\lambda\}$ and $\{\tau_\lambda\}$ are semi-proper, that is, they cover V and W, respectively, and, in the analytic case, there are compact subsets $K_\lambda \subset V_\lambda$, $L_\lambda \subset W_\lambda$, such that $\bigcup \sigma_\lambda(K_\lambda)$ and $\bigcup \sigma_\lambda(L_\lambda)$ are neighbourhoods of $\mathfrak{a}$ and $\mathfrak{b}$, respectively;*

3. *each Φ_λ is a monomial morphism (i.e., locally of the form* (4.4)*, where $D_\lambda = \{u_1 \cdots u_r = 0\}$ and $E_\lambda = \{x_1 \cdots x_p \cdot y_1 \cdots y_q = 0\}$).*

There are two important remarks about the previous results:

First, in the general **quasianalytic category**, the use of power substitutions is necessary because of the lack of the *extension property* discussed in §3.7. Indeed, example [BdSB19, Example 1.18] shows that it is, in general, impossible to monomialize a quasianalytic morphism without using power substitutions. In particular, the non-extension property constitutes an important technical difficulty in [BdSB19], which we overcome by using quasianalytic continuation (see §5.6) in a compatible way with resolution of singularities adapted to log-differentials (see §4.6).

Second, in the **algebraic and analytic categories**, our Theorem 4.4.5 has been previously proven by Cutkosky [Cu99, Cu05, Cu17]. The latter establish local monomialization along a valuation. At the one hand, we use étale-local blowings-up in the algebraic case, rather than Zariski-local blowings-up as in [Cu99, Cu05] – indeed Cutkosky shows that (along a valuation) the stronger result follows from the étale version. We plan to provide the necessary adaptations in a follow-up work (and we focused on the quasianalytic category in [BdSB19]). At the other hand, our approach provides a winning strategy for a version of Hironaka's game in the context of monomialization. More precisely, we provide a winning strategy for Alice in the following game that she plays against Bob:

- Start of the game: Bob chooses a morphism $\Phi_1 : (M_1, E_1) \to (N_1, D_1)$ and a point $\mathfrak{a}_1 \in M_1$;
- Alice's k-move: Alice chooses a smooth center $\mathcal{C}_k$ of blowing-up either in M_k or in N_k which are compatible with E_k of D_k, that is, she chooses a diagram:

$$\begin{array}{ccc} (M_{k+1}, D_{k+1}) & \xrightarrow{\sigma_k} & (M_k, D_k) \\ \downarrow{\scriptstyle \Phi_{k+1}} & & \downarrow{\scriptstyle \Phi_k} \\ (N_{k+1}, E_{k+1}) & \xrightarrow{\tau_k} & (N_k, E_k) \end{array}$$

where one of the morphisms σ_k or τ_k is a blowing-up with smooth center $\mathcal{C}_k$, while the other is the identity; and Φ_{k+1} is a well-defined morphism.

- Bob's k-move: Bob chooses any point $\mathfrak{a}_{k+1} \in \sigma_k^{-1}(\mathfrak{a}_k)$ where Φ_{k+1} is not monomial.
- End of the game: the game finishes at move k if Bob cannot make a move. In other words, the morphism $\Phi_{k+1} : (M_{k+1}, D_{k+1}) \to (N_{k+1}, E_{k+1})$ is monomial everywhere over $\sigma_k^{-1}(\mathfrak{a}_k)$.

Finally, in order to prove Theorems 4.4.4 and 4.4.5 we need to improve previous techniques of resolution of singularities adapted to log-differentials. In particular, we need to prove an intermediate result about a local relative monomialization of ideal sheaves, which is of independent interest:

Theorem 4.4.6 (Belotto da Silva, Bierstone [BdSB19])**.** *Let $\Phi : (M, D) \to (N, E)$ denote a monomial morphism of class $\mathcal{Q}$ and let $\mathcal{I}$ denote an ideal sheaf on M (we assume that the stalk of $\mathcal{I}$ at any point is generated by local sections defined in a common neighbourhood, though we do not assume that $\mathcal{I}$ is finitely generated; see Definition 3.6.3). Then there is a countable family (finite, in the algebraic case) of commutative diagrams* (4.6)*, where conditions (1), (2), (3) of Theorem 4.4.4 (or of Theorem 4.4.5, in the analytic and algebraic cases) hold, together with the following:*

(4) for each λ, the pull-back $\sigma_\lambda^*(\mathcal{I})$ is principal and monomial.

Recently, I got in contact with the works of Abramovich, Temkin and Włodarczyk [ATW20a] [ATW20b] about relative desingularization and principalization of ideals for logarithmically regular morphisms, and their application to reduction of morphisms. Their main result provides a monomialization of dominant algebraic morphisms in characteristic zero via more general blowings-up, including the blow up of fractional powers of monomial ideals. Following personal communication with the authors, I understand that it would be very interesting to make a careful comparison between their technique with the methods we use in [BdSB19]. For now, I am only confortable remarking that there are definitely some similar ideas.

4.5 Logarithmic Fitting ideals

Let (M, D) denote a real or complex analytic manifold with SNC divisor. We denote by $\Omega^1_M(-\log D)$ the sheaf of modules over the structure sheaf $\mathcal{O}_M$ of logarithmic differential 1-forms, following Saito [S80]. In coordinates

$$(\boldsymbol{u}, \boldsymbol{w}) = (u_1, \ldots, u_r, w_1, \ldots, w_t) \tag{4.7}$$

compatible with D at a point $\mathfrak{a} \in M$ (where $D = \{u_1 \cdots u_r = 0\}$), the local sections of $\Omega^1_M(-\log D)$ are generated by

$$\frac{du_1}{u_1}, \ldots, \frac{du_r}{u_r}, \, dw_1, \ldots, dw_t.$$

The sheaf $\Omega^k_M(-\log D)$ of logarithmic k-forms over M is defined in terms of $\Omega^1_M(-\log D)$ in the standard way.

Consider an analytic morphism $\Phi : (M, D) \to (N, E)$ such that $\Phi^{-1}(E)$ is a SNC divisor contained in D. Let $n = \dim N$ and $m = \dim M$.

Lemma 4.5.1 (Pullbacks of log differentials are well-defined [BdSB19, Lemma 4.1])**.** *If $\Phi^{-1}(E)$ is a SNC divisor contained in D, then the pullback $\sigma^* \left(\Omega^k_N(-\log E)\right)$ is a subsheaf of $\Omega^k_M(-\log D)$ for all $k = 1, \ldots, n$.*

Definition 4.5.2 (Logarithmic Fitting Ideals [BdSB19, Definition 4.2])**.** *The* logarithmic Fitting ideal sheaf $\mathcal{F}_{n-k}(\Phi)$ *associated to Φ is the ideal subsheaf of $\mathcal{O}_M$ whose stalk $\mathcal{F}_{n-k}(\Phi)_\mathfrak{a}$ at $\mathfrak{a} \in M$ can be described (in coordinates* (4.7) *at $\mathfrak{a}$) in the following way. If ω be a logarithmic k-form at $b = \Phi(a)$, then*

$$\Phi^*\omega = \sum_{I,J} B^\omega_{I,J}(\boldsymbol{u}, \boldsymbol{w}) \frac{du_{i_1}}{u_{i_1}} \wedge \cdots \wedge \frac{du_{i_l}}{u_{i_l}} \wedge dw_{j_1} \wedge \cdots \wedge dw_{j_{k-l}},$$

where the sum is over all pairs (I, J) with $I = (i_1, \ldots, i_l)$, $1 \leq i_1 < \cdots < i_l \leq r$, and $J = (j_1, \ldots, j_{k-l})$, $1 \leq j_1 < \cdots j_{k-l} \leq t$, and where the coefficients $B^\omega_{I,J}$ are analytic germs at $\mathfrak{a}$, by Lemma 4.5.1. Then $\mathcal{F}_{n-k}(\Phi)_a$ is generated by the set of coefficients $B^\omega_{I,J}(\boldsymbol{u}, \boldsymbol{w})$, for all $\omega \in \Omega^k_{N,b}(-\log E)$ and all (I, J).

We refer the reader to the work of Teissier [Te77, §1] for the classical notion of Fitting Ideal and its relationship with morphisms.

Logarithmic Fitting Ideals can be used to characterize “monomial” normal forms from a morphism. Indeed, we are able to characterize both Hsiang-Pati coordinates (see Conjecture 4.2.1) and monomial morphism (see Definition 4.4.1) via log-Fitting ideals:

Theorem 4.5.3 (Characterization of Hsiang-Pati coordinates [BdSBGM17, Lemma 3.1, Theorem 1.2])**.** *Let $\Phi : (M, D) \to (N, \emptyset)$ be an analytic morphism between smooth varieties. Let $\mathfrak{a} \in M$. Then, for each $k \in \{1, \ldots, m = \dim M\}$ the following are equivalent:*

1. *The logarithmic Fitting ideals* $\mathcal{F}_{m-1}(\Phi), \ldots, \mathcal{F}_{m-k}(\Phi)$ *are all principal monomial ideals at* $\mathfrak{a}$.

2. *There are analytic (or étale) coordinates* $(\boldsymbol{u}, \boldsymbol{v})$ *centred at* $\mathfrak{a}$ *and compatible with* E, *and coordinates* $\boldsymbol{z} = (z_1, \ldots, z_N)$ *of* N *at* $\Phi(\mathfrak{a})$, *such that, writing* $\Phi = (\phi_1, \ldots, \phi_n)$ *with respect to the coordinates* $\boldsymbol{z}$,

   a) *the submodule* $\mathcal{M}_k$ *of* $\Omega^1_{M,\mathfrak{a}}$ *generated by the pull-backs* $\sigma^* dz_m = d\sigma_m$, $m = 1, \ldots, k$, *is also generated by differential monomials*

   $$d(\boldsymbol{u}^{\alpha_i}), \quad i = 1, \ldots, l_k \quad \text{and} \quad d(\boldsymbol{u}^{\beta_j} v_j), \quad j = 1, \ldots, k - l_k$$

   for some $l_k \leq k$, *such that* $\alpha_1 \wedge \ldots \wedge \alpha_{l_k} \not\equiv 0$ *and the set* $\{\alpha_i, \beta_j\}$ *is totally ordered;*

   b) *for each* $r > k$, $\Phi_r = g_r + S_r$, *where* $dg_r \in \mathcal{M}_k$ *and* S_r *is divisible by* $\boldsymbol{u}^{\max\{\boldsymbol{\alpha}_{l_k}, \boldsymbol{\beta}_{k-l_k}\}}$.

In the case that all non-trivial logarithmic Fitting Ideals are principal monomial, we call the coordinates $(\boldsymbol{u}, \boldsymbol{v})$ *a Hsiang-Pati coordinates.*

The following is a version of the *rank theorem* for logarithmic derivatives; cf. [O18, Ch. IV]:

Theorem 4.5.4 (Characterization of Monomial morphisms [BdSB19, Theorem 4.4])**.** *A dominant* $\mathcal{Q}$*-morphism* $\Phi : (M, D) \to (N, E)$ *is monomial at a point* $\mathfrak{a} \in M$ *if and only if the log Fitting ideal* $\mathcal{F}_0(\Phi)_{\mathfrak{a}}$ *is generated by a unit.*

4.6 Resolution of singularities adapted to log-differentials

We present some of the main general ideas behind our techniques of resolution of singularities adapted to log-differentials, by combining the presentation of several papers [BdS16b, BdS18, BdSB19]. We note that the development of these methods are strongly influenced by the works of Bierstone and Milman [BM08], Cutkosky [Cu02] and Denkowska and Roussarie [DR91].

We consider a pair (M, E), where M is an analytic manifold and E is a SNC divisor, a coherent sub-sheaf $\Delta \subset Der_M(-\log E)$ (following Saito [S80]) and a coherent ideal sheaf $\mathcal{I} \subset \mathcal{O}_M$. Our goal is to develop techniques of resolution of singularities of $\mathcal{I}$ which are "compatible" with the distribution Δ. As a concrete example of a possible application, we may consider the problem of resolving the singularities of $\mathcal{I}$ while preserving a certain class of singularities of Δ (e.g. simple, log-canonical, monomial, etc); see Problem 2 below.

Chain of Ideals and log differential order: We start by defining our main invariant, which replaces the *order* in the usual resolution of singularities algorithms. We denote by $\Delta(\mathcal{I})$ the coherent ideal sheaf whose stalk at each point $\mathfrak{a} \in M$ is

$$\Delta(\mathcal{I})_{\mathfrak{a}} := \{\partial(f) : f \in \mathcal{I}_{\mathfrak{a}},\, \partial \in \Delta_{\mathfrak{a}}\}.$$

If $f \in \mathcal{O}_M$, we also write $\Delta(f) := \Delta(\mathcal{I})$, where $\mathcal{I}$ is the principal ideal generated by f.

Definition 4.6.1. *(Closure by repeated derivatives, and log differential order [BdSB19, Def. 5.1]) Given a coherent ideal sheaf $\mathcal{I}$ and a coherent submodule Δ of $Der_M(-\log E)$, we define a chain of coherent ideals,*

$$\mathcal{I} = \mathcal{I}_0^{\Delta} \subset \mathcal{I}_1^{\Delta} \subset \mathcal{I}_2^{\Delta} \subset \cdots \subset \mathcal{I}_k^{\Delta} \subset \cdots , \qquad (4.8)$$

where $\mathcal{I}_{k+1}^{\Delta} = \mathcal{I}_k^{\Delta} + \Delta(\mathcal{I}_k^{\Delta})$, $k = 0, 1, \ldots$, and we define the closure *of $\mathcal{I}$ by Δ as the ideal*

$$\mathcal{I}_{\infty}^{\Delta} := \sum_{k=0}^{\infty} \mathcal{I}_k^{\Delta}.$$

We say that an ideal sheaf $\mathcal{I}$ is Δ-closed if $\mathcal{I} = \mathcal{I}_{\infty}^{\Delta}$.

Given $\mathfrak{a} \in M$, we define the log differential order $\mu_{\mathfrak{a}}(\mathcal{I}, \Delta)$ *of $\mathcal{I}$* relative to Δ *as the smallest $\mu \in \mathbb{N} \cup \{\infty\}$ such that $\mathcal{I}_{\mu}^{\Delta} \cdot \mathcal{O}_{\mathfrak{a}} = \mathcal{I}_{\infty}^{\Delta} \cdot \mathcal{O}_{\mathfrak{a}}$. (By convention, $\mu_{\mathfrak{a}}(\mathcal{I}, \Delta) := \infty$ if $\mathcal{I} = 0$.)*[3]

Note that if $\Delta = Der_M$, then the log-differential order coincides with the order of the ideal $\mathcal{I}$, and its Δ-closure $\mathcal{I}_{\infty}^{\Delta}$ is always equal to the structural ring $\mathcal{O}_M$. Another preliminary interesting case happens when $\Delta = Der_M(-\log E)$, in which case the Δ-closure $\mathcal{I}_{\infty}^{\Delta}$ is necessarily a principal monomial ideal, see Lemma 4.6.6 below and c.f. [BM08]. Dealing with an ideal sheaf whose Δ-closure $\mathcal{I}_{\infty}^{\Delta}$ is *not* principal monomial is an important extra technical difficulty, which makes our treatment different from usual resolution of singularities algorithms.

[3]Since $\mathcal{O}_{\mathfrak{a}}$ are Noetherian rings, this invariant is always finite and $\mathcal{I}_{\infty}$ is always coherent. This is not known to be true in the quasianalytic case, where local ideals are not known to be Noetherian in general.

We finish this paragraph by stating some useful properties of log-differential order, which easily follow from the definition:

Lemma 4.6.2. *(Properties of closure by Δ [BdSB19, Lemma 5.3])Let $\mathcal{I}$, $\mathcal{J}$ denote coherent ideals, and Δ, Δ_1, Δ_2 coherent submodules of $Der_M(-\log E)$.*

1. *If $\mathcal{I} \subset \mathcal{J}$, then $\mathcal{I}_\infty^\Delta \subset \mathcal{J}_\infty^\Delta$.*
2. *If $\Delta_1 \subset \Delta_2$, then $\mathcal{I}_\infty^{\Delta_1} \subset \mathcal{I}_\infty^{\Delta_2}$.*
3. *Suppose that $\mathcal{I}_\infty^\Delta$ is principal and monomial (with respect to D) at $\mathfrak{a} \in M$. If $\mu_a(\mathcal{I}, \Delta) > 0$, then there exists a regular vector field $\partial \in \Delta_\mathfrak{a}$.*

Monomial singular distributions: Let $\mathcal{K}$ be a subfield of $\mathbb{C}$. We now introduce a class of singularities of singular distributions Δ, inspired by [DR91], which plays an important role in our works about first integrals and monomialisation of morphisms, see §4.3 and 4.4; and also in our work about the cotangent sheaf (although we didn't state it explicitly), see §4.2.

Definition 4.6.3. *(Monomial singular distribution [BdS18, Lemma 3.4]) A coherent and involutive singular distribution $\Delta \subset Der_M(-\log D)$ of leaf dimension d is said to be $\mathcal{K}$-monomial at $\mathfrak{a} \in M$ if there exists coordinate systems $\boldsymbol{x}$ adapted to D, and a $m-d$ complex multi-valued monomials $(\boldsymbol{x}^{\beta_1}, \ldots, \boldsymbol{x}^{\beta_{m-d}})$ with $\beta_i \in \mathcal{K}^m$ for $i = 1, \ldots, m-d$, and $\beta_1 \wedge \ldots \wedge \beta_{m-d} \not\equiv 0$, such that:*

$$\Delta_\mathfrak{a} = \{\partial \in Der_p\mathfrak{a}(-\log D);\ \partial(\boldsymbol{x}^{\beta_i}) \equiv 0 \text{ for } i = 1, \ldots, m-d\}$$

We say that Δ is $\mathcal{K}$-monomial if it is $\mathcal{K}$-monomial at every point.

The question of the beginning of the section can now be made precise:

Problem 2. *Given an ideal sheaf $\mathcal{I}$ and $\mathcal{K}$-monomial singular distribution Δ, is it possible to find a resolution of singularities $\sigma : (\widetilde{M}, \widetilde{E}) \to (M, E)$ of $\mathcal{I}$ in such a way that the strict transform $\widetilde{\Delta}$ of Δ is $\mathcal{K}$-monomial?*

This is the simplest (non-trivial) problem which we can address via our techniques, see [BdS16b], and will serve as a guiding thread throughout this section. Let us now discuss some useful properties of $\mathcal{K}$-monomial distribution. Note that the notion of a $\mathcal{K}$-monomial distribution is open that is, if Δ is $\mathcal{K}$-monomial at a point $\mathfrak{a}$, then it also has this property at a neighborhood U of $\mathfrak{a}$ [BdS18, Lemma 3.6]. We now establish a useful set of local generators for $\mathcal{K}$-distributions:

Lemma 4.6.4. *(Log derivatives tangent to a monomial distribution, [BdS18, Def. 3.1], [BdSB19, Lemma 5.9]) Suppose that Δ is a $\mathcal{K}$-monomial singular distribution and fix $\mathfrak{a} \in M$. There exists an adapted coordinate system $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = (u_1, \dots, u_r, v_1, \dots, v_s, w_1, \dots, w_t)$ at $\mathfrak{a}$, which we call a monomial coordinate system, such that $\Delta_{\mathfrak{a}}$ is generated by*

$$Y^j := \sum_{i=1}^{r} \gamma_{ji} u_i \frac{\partial}{\partial u_i}, \quad j = 1, \dots, r-p, \quad \textit{and} \quad Z^l := \frac{\partial}{\partial w_l}, \quad l = 1, \dots, t, \tag{4.9}$$

where $\boldsymbol{\gamma}_j = (\gamma_{j1}, \dots, \gamma_{jr}) \in \mathcal{K}^r$, $j = 1, \dots, r-p$, form a basis of the orthogonal complement of the $\mathcal{K}$-linear subspace spanned by $\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_p$ with respect to the standard scalar product $\langle \boldsymbol{\gamma}, \boldsymbol{\alpha} \rangle$.

Next, we note that functions and ideal sheaves admit a Taylor expansion which is *compatible* with $\mathcal{K}$-monomial distributions. More precisely:

Lemma 4.6.5. *(Formal eigenvectors [BdS16b, Lemma 3.7], [BdSB19, Lemma 5.13]; cf. [Cu99, §7]) Let Δ be a $\mathcal{K}$-monomial singular distribution over (M, D), and fix a point $\mathfrak{a} \in M$. In the coordinate system of Lemma 4.6.4, given a function $f \in \mathcal{O}_{\mathfrak{a}}$, the formal Taylor expansion $\hat{f}_{\mathfrak{a}}(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = T_{\mathfrak{a}} f(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$ of f with respect to the coordinates $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$ can be written*

$$\widehat{f}_{\mathfrak{a}}(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = \sum_{\boldsymbol{\delta} \in \mathbb{N}^t} \boldsymbol{w}^{\boldsymbol{\delta}} \sum_{\boldsymbol{\lambda} \in \mathcal{K}^{r-p}} \widehat{f}_{\boldsymbol{\delta}\boldsymbol{\lambda}}(\boldsymbol{u}, \boldsymbol{v}), \tag{4.10}$$

where there are only a countable number of indices $(\boldsymbol{\lambda}, \boldsymbol{\delta})$ such that $\widehat{f}_{\boldsymbol{\delta}\boldsymbol{\lambda}}(\boldsymbol{u}, \boldsymbol{v}) \neq 0$, and the $\widehat{f}_{\boldsymbol{\delta},\boldsymbol{\lambda}}$ are eigenvectors of the set of generators $\{Y^1, \dots, Y^{r-p}, Z_1, \dots, Z^t\}$ of Δ_a^{Φ}; i.e.,

$$Y^i(\widehat{f}_{\boldsymbol{\delta}\boldsymbol{\lambda}}(\boldsymbol{u}, \boldsymbol{v})) = \lambda_i \widehat{f}_{\boldsymbol{\delta}\boldsymbol{\lambda}}(\boldsymbol{u}, \boldsymbol{v}), \quad i = 1, \dots, r-p+t, \quad \lambda_i \in \mathcal{K}.$$

In particular, the above result implies that Δ-closed ideal sheaves admit a system of generators of eigenvectors of Δ, that is:

Lemma 4.6.6. *(Formal generators of a Δ-closed ideal, [BdS16b, Corollary 3.8], [BdSB19, Lemma 5.13]) Let Δ be a $\mathcal{K}$-monomial singular distribution over (M, D), and fix a point $\mathfrak{a} \in M$. Let $\mathcal{I}$ be a coherent ideal sheaf which is closed by Δ; i.e., $\Delta(\mathcal{I}) \subset \mathcal{I}$. In the notation of Lemma 4.6.4, the ideal $\mathcal{I}_{\mathfrak{a}}$ admits a finite system of generators $\{f_{\boldsymbol{\lambda}}(\boldsymbol{u}, \boldsymbol{v})\}$, where each $f_{\boldsymbol{\lambda}}(\boldsymbol{u}, \boldsymbol{v})$ is an eigenvector of $\{Y^1, \dots, Y^{r-p}, Z^1, \dots, Z^t\}$. In particular, $\mathcal{I}_{\mathfrak{a}}$ admits a system of generators that are independent of the variable $\boldsymbol{w}$.*[4]

[4]The proof of the above Lemma relies on the Noetherianity of $\mathcal{O}_{\mathfrak{a}}$. In the quasianalytic category, where local rings are not-known to be Noetheiran, we can only prove the result formally, see [BdSB19, Lemma 5.13]. This constitutes an important technical issue in [BdSB19].

Interlude: Log-derivatives tangent to a morphism [BdSB19] Let us briefly recall the notion of log-derivatives tangent to a morphism defined in [BdSB19], and discuss its relation with monomial distributions:

Definition 4.6.7. *(Log-derivatives tangent to a morphism [BdSB19, Def. 5.7]) Let $\Phi : (M, D) \to (N, E)$ denote a morphism of analytic manifolds with SNC divisors. The sheaf of* log derivatives tangent to Φ *(i.e., tangent to the fibres of Φ) is the sheaf of $\mathcal{O}_M$-submodules $\Delta^{\Phi} \subset Der_M(-\log D)$ whose stalk at each $\mathfrak{a} \in M$ is*

$$\Delta^{\Phi}_{\mathfrak{a}} := \{\partial \in \mathrm{Der}_{\mathfrak{a}}(-\log D) : \partial(f \circ \Phi) = 0,\ f \in \mathcal{O}_{\Phi(\mathfrak{a})}\}.$$

The notion of log-derivatives tangent to a morphism coincide with the notion of *relative log derivations* of origin in work of Grothendieck and Deligne, see [O18, Ch. IV].

If Φ is a monomial morphism (see Definition 4.4.1), then Δ^{Φ} is a $\mathbb{Q}$-monomial distribution. But the converse is not true, that is, Δ^{Φ} may be $\mathbb{Q}$-monomial even if Φ is not monomial. Keeping track of the monomial form of a morphism demands extra considerations about the target of the morphism, which we do not discuss in this memoir. We refer the reader to [BdSB19, §5, 6 and 7] for details.

Blowings-up and the log-differential order: The log-differential order introduced in Definition 4.6.1 may increase under general blowings-up, as is illustrated by the following example:

Exemple 4.6.7.1. *[BdS16b, Ex. 1.8] Let $n > 2$ be a natural number, and consider the ideal sheaf $\mathcal{I} = (y^2 + xz^n + x^{n+1})$ and the singular distribution $\Delta = (\partial_y, \partial_z)$ defined in $\mathbb{C}^3$ (which may represent, for example, the log-relative derivations in respect to the linear projection $\pi(x, y, z) = x$). We easily compute that:*

$$\mathcal{I}_1^{\Delta} = (y, xz^{n-1}, x^{n+1}), \quad \mathcal{I}_2^{\Delta} = (1)$$

which implies that the log-differential order $\mu_{\mathfrak{a}}(\mathcal{I}, \delta) = 2$ at every point $\mathfrak{a} \in Z(\mathcal{C})$ where $\mathcal{C} = Z(x, y)$. Now, let $\sigma : (M, E) \to (\mathbb{C}^3, \emptyset)$ be the blowing-up with center $\mathcal{C}$ and let us consider the x-chart:

$$x = u, \quad y = uv, \quad z = z$$

and we compute the pull-back of $\mathcal{I}$ and the weak (or strict) transform of Δ in a simple way:

$$\widetilde{\mathcal{I}} = \sigma^*(\mathcal{I}) = u(v^2 u + z^n + u^n)$$

$$\widetilde{\Delta} = \sigma^*(\Delta) \cap Der_{\mathbb{C}^3} = \left(\frac{1}{u}\partial_v, \partial_z\right) \cap Der_{\mathbb{C}^3} = (\partial_v, \partial_z).$$

In particular, note that $\sigma^*(\Delta) \neq \widetilde{\Delta}$. *It now follows that:*

$$\widetilde{\mathcal{I}}_1^{\widetilde{\Delta}} := u(vu, z^{n-1}, u^n), \quad \widetilde{\mathcal{I}}_2^{\widetilde{\Delta}} = u(u, z^{n-2}) \quad \ldots \quad \widetilde{\mathcal{I}}_{n-1}^{\widetilde{\Delta}} = u(u, z), \quad \widetilde{\mathcal{I}}_n^{\widetilde{\Delta}} = (u)$$

which implies that the log-differential order $\mu_{\mathfrak{a}}(\widetilde{\mathcal{I}}, \widetilde{\Delta}) = n > 2$ *at every point* $\mathfrak{a} \in Z(u, z)$.

One of the main issues illustrated by the above example is that the pull-back of the generators of Δ are *asymmetric*, that is, one of them has a pole while the other does not. In particular, we have that $\sigma^*(\Delta) \neq \widetilde{\Delta}$. This phenomenon changes the "weight" that each derivation plays in the computation of the log-differential order; more precisely, in the example the derivation ∂_y dominates the computation of the invariant before blowing-up in the sense that $\partial_y^2(\mathcal{I}) = (1)$, while after blowing-up the pull-back of ∂_y does not intervene in the computation, which essentially depends on the pull-buck of ∂_z. In contrast, if we do have that $\sigma^*(\Delta) = \widetilde{\Delta}$, then this phenomenon is excluded, and we can control the log-differential order:

Lemma 4.6.8. *(Pullback of the ideal chain by blowings-up and power substitutions, [BdSB19, Lemma 5.3]) Let* $\mathcal{I} \subset \mathcal{O}_M$ *denote a coherent ideal and* Δ *be a coherent submodule of* $Der_M(-logE)$. *Let* $\sigma : (\widetilde{M}, \widetilde{E}) \to (M, E)$ *denote the composite of a finite sequence of (local) blowings-up and (local) power substitutions compatible with* E. *Suppose that* $\sigma^*(\Delta) = \widetilde{\Delta}$, *that is,* $\sigma^*(\Delta)$ *is a well-defined* $\mathcal{O}_{\widetilde{M}}$*-submodule of* $Der_{\widetilde{M}}(-log\widetilde{E})$. *Let* $\widetilde{\mathcal{I}} = \sigma^*(\mathcal{I})$ *denote the total transform. Then*

1. $\sigma^*(\mathcal{I}_k^{\Delta}) = \widetilde{\mathcal{I}}_k^{\widetilde{\Delta}}$, *for all* $k \in \mathbb{N} \cup \{\infty\}$;
2. *for every* $\widetilde{\mathfrak{a}} \in \widetilde{M}$, $\mu_{\widetilde{\mathfrak{a}}}(\widetilde{\mathcal{I}}, \widetilde{\Delta}) \leq \mu_{\sigma(\widetilde{\mathfrak{a}})}(\mathcal{I}, \Delta)$.

In general, it is impossible to resolve singularities of $\mathcal{I}$ via a sequence of blowings-up σ such that $\sigma^*(\Delta) = \widetilde{\Delta}$. It is therefore necessary to deal with the phenomena illustrated in Example 4.6.7.1. We do this via two complementing ideas. First, by restricting the class of blowings-up that are allowed to the so called Δ-admissible blowings-up. Second, by working with Weierstrass-Tschirnhausen normal forms adapted to the distribution Δ, which allow us to control the asymmetries appearing in the pull-back of the distribution Δ via an induction procedure.

Roughly, the first idea is used to reduce the case of a general ideal sheaf $\mathcal{I}$ to the case of an ideal sheaf $\widetilde{\mathcal{I}}$ whose Δ-closure $\widetilde{\mathcal{I}}_{\infty}^{\Delta}$ is principal monomial. The second idea is used in order to reduce the log-differential order of an ideal sheaf $\widetilde{\mathcal{I}}$ whose Δ-closure $\widetilde{\mathcal{I}}_{\infty}^{\Delta}$ is already principal monomial.

Δ-admissible blowings-up: Given an ideal sheaf $\mathcal{I}$, we consider ideal sheaves $\Gamma_{\Delta,k}(\mathcal{I})$, which we call *generalized k-Fitting ideal sheaves*, whose stalk at each point $\mathfrak{a}$ in M is generated by all terms of the form:

$$det \begin{Vmatrix} \partial_1(f_1) & ... & \partial_1(f_k) \\ \vdots & \ddots & \vdots \\ \partial_k(f_1) & ... & \partial_k(f_k) \end{Vmatrix}$$

for $\partial_i \in \Delta \cdot \mathcal{O}_{\mathfrak{a}}$ and $f_j \in \mathcal{I}_{\mathfrak{a}}$.

We recall that classical Fitting ideals are computed in respect to the sheaf of derivations Der_M, c.f. [Te77, §1]. We say that $\Gamma_{\Delta,k}(\mathcal{I})$ are "generalized" Fitting ideals in the sense that we replace Der_M by an arbitrary coherent sheaf of derivations Δ.

Definition 4.6.9 (Δ-admissible blowing-up)**.** *Consider a blowing-up $\sigma : (\widetilde{M}, \widetilde{E}) \to (M, E)$ compatible with E and let $\mathcal{I}_{\mathcal{C}}$ be the ideal sheaf of the center of blowings-up. We say that σ is Δ-admissible if there exists $d_0 \in \mathbb{N}$ such that:*

1. *The generalized k-Fitting ideal sheaf $\Gamma_{\Delta,k}(\mathcal{I}_{\mathcal{C}})$ is equal to $\mathcal{O}_M$ for $k \leq d_0$;*
2. *The ideal sheaf $\Gamma_{\Delta,k}(\mathcal{I}_{\mathcal{C}}) + \mathcal{I}_{\mathcal{C}}$ is equal to $\mathcal{I}_{\mathcal{C}}$ for $k > d_0$.*

We say that the blowing up is Δ-invariant if moreover $d_0 = 0$.

One of the first interesting properties of Δ-admissible blowings-up is that they preserve the monomiality of a singular distribution, that is:

Proposition 4.6.10 (Belotto da Silva, [BdS16b, Prop. 4.4])**.** *Let Δ be a $\mathcal{K}$-monomial singular distribution in (M, E), and $\sigma : (\widetilde{M}, \widetilde{E}) \to (M, E)$ be a Δ-admissible blowing-up. Then the weak and strict transforms of Δ coincide, that is, $\widetilde{\Delta} = \Delta^{st} = \Delta^{w}$ (see §3.5), and are $\mathcal{K}$-monomial.*

Now, it is easy to see that if $\sigma : (\widetilde{M}, \widetilde{E}) \to (M, E)$ is a Δ-invariant blowing-up, then $\sigma^*(\Delta)$ is a sub-sheaf of $Der_{\widetilde{M}}(-\log \widetilde{E})$, that is, there are no poles [BdS16b, Lemma 7.3]. In particular, since the singular distributions Δ are all tangent to the exceptional divisor E, all combinatorial blowings-up are Δ-admissible, c.f. [BdSB19, Cor.5.19]. Finally, by combining Lemma 4.6.8 and the functoriality of resolution of singularities, we derive the following result:

Proposition 4.6.11 (Δ-invariant reduction, [BdS16b, Prop. 6.1], c.f. [BdSB19, Assertion I.A.b]). *Let M be an analytic manifold, E be a SNC divisor on it, Δ be an involutive singular distribution and $\mathcal{I}$ be a coherent ideal sheaf. Let M_0 be a relatively compact open set of M, and let $E_0 = E \cap M_0$, $\mathcal{I}_0 = \mathcal{I} \cdot \mathcal{O}_{M_0}$ and $\Delta_0 = \Delta \cdot \mathcal{O}_{M_0}$. There exists a sequence of Δ-invariant blowings-up:*

$$(\widetilde{M}, \widetilde{\Delta}, \widetilde{\mathcal{I}}, \widetilde{E}) = (M_r, \Delta_r, \mathcal{I}_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_1} (M_0, \Delta_0, \mathcal{I}_0, E_0)$$

where $\mathcal{I}_{i+1} = \sigma_i^*(\mathcal{I}_i)$ and $\Delta_{i+1} = \sigma^w(\Delta_i)$ denotes the weak transform of Δ_i, for every $i = 0, \ldots, r-1$, such that $\widetilde{\mathcal{I}}_\infty^\Delta$ is a principal monomial ideal sheaf and, at every point $\mathfrak{b} \in \widetilde{M}$, we have that $\mu_{\mathfrak{b}}(\widetilde{\mathcal{I}}, \widetilde{\Delta}) \leq \mu_{\sigma(\mathfrak{b})}(\mathcal{I}, \Delta)$. In particular, if Δ is $\mathcal{K}$-monomial, then $\widetilde{\Delta}$ is $\mathcal{K}$-monomial.

Weierstrass-Tschirnhausen normal forms When the distribution Δ is $\mathcal{K}$-monomial and the closure ideal sheaf $\mathcal{I}_\infty^\Delta$ is principal monomial, we can derive more precise normal forms than Lemma 4.6.4; in particular, we are able to make a Tschirnhausen transform:

Lemma 4.6.12 (Weierstrass-Tschirnhausen normal form, see [BdSB19, Lemma 5.22], [BdS16b, Lemma 5.2] and [BdSB19, Lemma 5.22]). *Let Δ be a $\mathcal{K}$-monomial singular distribution over (M, E), and $\mathcal{I}$ be a coherent ideal sheaf on M. Let $\mathfrak{a} \in M$.*

1. *The ideal $\mathcal{I}_{\mathfrak{a}}$ is principal and monomial if and only if $\mathcal{I}_{\infty,\mathfrak{a}}^\Delta := \mathcal{I}_\infty^\Delta \cdot \mathcal{O}_{\mathfrak{a}}$ is principal and monomial, and $\mu_{\mathfrak{a}}(\mathcal{I}, \Delta) = 0$.*

2. *If $\mathcal{I}_{\infty,\mathfrak{a}}^\Delta$ is principal and monomial and $d := \mu_{\mathfrak{a}}(\mathcal{I}, \Delta) > 0$, then there exists a monomial coordinate system $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = (\boldsymbol{u}, \boldsymbol{v}, w_1, \widehat{\boldsymbol{w}}_1)$ at $\mathfrak{a}$ (see Lemma 4.6.4), such that $\mathcal{I}_{\mathfrak{a}}$ has a set of generators $\{F_\iota\}_{\iota \in I}$ of the form*

$$F_\iota(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = \boldsymbol{u}^{\boldsymbol{\gamma}} \left\{ \widetilde{F}_\iota(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) w_1^d + \sum_{j=0}^{d-1} f_{\iota j}(\boldsymbol{u}, \boldsymbol{v}, \widehat{\boldsymbol{w}}_1) w_1^j \right\},$$

   where $\mathcal{I}_{\infty,\mathfrak{a}} = (\boldsymbol{u}^{\boldsymbol{\gamma}})$ and there exists $\iota \in I$ such that $\widetilde{F}_\iota(0) \neq 0$, $f_{\iota,d-1} = 0$ and $f_{\iota j}(0) = 0$, $j = 0, \ldots, d-2$.

Variations of the above normal form can be found in: [BdS18, Lemma 5.3] for first integrals, [BdSBGM17, Lemma 4.3] for Fitting ideals, [BdSB19, Lemma 5.29] for morphisms. The normal form given in Lemma 4.6.12 and

Proposition 4.6.11 are used in order to build up the induction scheme presented in [BdS16b, § 6] (originally inspired by the work [Cu02] of Cutosky) which proves the following technical result:

Theorem 4.6.13 (Belotto da Silva, [BdS16b])**.** *Let M be an analytic manifold, E be a SNC divisor, Δ be an involutive singular distribution and $\mathcal{I}$ be a coherent ideal sheaf. Let $\mathfrak{a} \in M$. Then there is an open neighbourhood V of $\mathfrak{a}$ and a finite number of morphisms*

$$\sigma_\lambda : (V_\lambda, E_\lambda) \to (M, E) \tag{4.11}$$

where

1. *each σ_λ is a composite of finitely many local Δ-admissible blowings-up;*
2. *the family of morphisms $\{\sigma_\lambda\}$ is semi-proper, that is, they cover V, and there are compact subsets $K_\lambda \subset V_\lambda$, such that $\bigcup \sigma_\lambda(K_\lambda)$ is a (compact) neighbourhood of $\mathfrak{a}$;*
3. *Every pull-back $\mathcal{I}_\lambda = \sigma_\lambda^*(\mathcal{I})$ is a principal and monomial ideal sheaf.*

In particular, if Δ is $\mathcal{K}$-monomial, then Δ_λ is $\mathcal{K}$-monomial, where Δ_λ denotes the strict transform of Δ by σ_λ.

Note that the above Theorem provides a local answer to Problem 2. We finish by noting that the induction procedure introduced in [BdS16b, §6] serves as a prototype for the induction procedure used in our future works, where more delicate objects are studied, c.f. [BdSBGM17, §5], [BdS18, §7] and [BdSB19, §7].

Chapter 5

Real quasianalytic geometry

5.1 Introduction

Even if the interest in quasianalytic classes goes back to the beginning of the XXth century, their algebraic and geometrical properties are far from being understood (see [Th08] for an overview). From a model-theoretical point of view, their structure is comparable to analytic functions : they are o-minimal, polynomially bounded and model complete [RSW03]. Nevertheless, several algebraic properties of analytic functions do not extend to quasianalytic classes, see §3.7. For example, there is no quasianalytic Weierstrass preparation Theorem [Chi76], [ABBNZ14], [PR13], and several important questions about division, factorization, composition, etc, are open. In particular, it is not known if all their local rings are Noetherian.

Since the beginning of the 2000's, the scenario has changed. Already in their original proof of a constructive resolution of singularities, Bierstone and Milman pointed out that a quasianalytic hypersurface admitted a resolution of singularities [BM97, p.208(3)]. Soon after, Rolin Speissegger and Wilkie proved that quasianalytic classes admit local resolution of singularities [RSW03] and, at the same time, Bierstone and Milman remarked that their proof of global resolution of singularities [BM97] extended, in an easy way, to quasianalytic classes [BM04]. Resolution of singularities is one of the few algebraic techniques which quasianalytic functions share with analytic functions. It is a strong tool to treat algebraic-geometric problems in quasianalytic geometry because, in certain situations, it can be used as a substitute to the Weierstrass preparation property, c.f. [RSW03, RS15, BdSB19].

First, in collaboration with Bierstone, we have explored resolution of singularities in order to study algebraic properties of quasianalytic classes. This led to our works with Biborski [BdSBB17] and Chow [BdSBC18], where we introduce the technique of *quasianalytic continuation*. This technique allows one to "propagate" to a neighborhood properties which are only formally verified at a point. This method allowed us to study regularity properties of solutions of quasianalytic equations and composition of quasianalytic functions; we provide details in §5.2 and 5.3 below. Recently, in collaboration with Bierstone and Kiro [BdSBK20], we investigated the limitations of resolution of singularities in quasianalytic geometry; see §5.4 below for details.

Second, in collaboration with Bierstone, by combining quasianalytic continuation with new methods of *resolution of singularities tangent to log-differentials* (see §4.6), we obtain a *monomialization of quasianalytic morphisms* [BdSB19], an extension of resolution of singularities to maps. As an application, we obtain a new proof of rectiliniarization of definable sets in $\mathbb{R}_{\mathcal{Q}}$, previously proved by Rolin and Servi using model-theoretical techniques [RS15], see §5.5.

Finally, we finish the chapter by providing an introduction to *quasianalytic continuation* in §5.6.

5.2 Solution of quasianalytic equations [BdSBB17]

In collaboration with Biborski and Bierstone [BdSBB17], we have developed techniques for solving equations

$$G(x,y) = 0, \text{ where } (x,y) = (x_1,\dots,x_n,y)$$

and G is a function in a given quasianalytic class. Assuming that $G(x,y) = 0$ has a formal power series solution $y = H(x)$ at some point $\mathfrak{a}$, we ask whether H is the Taylor expansion at $\mathfrak{a}$ of a quasianalytic solution $y = h(x)$, where $h(x)$ is allowed to have a certain controlled loss of regularity, depending on G. More precisely:

Theorem 5.2.1 (Belotto da Silva, Biborski, Bierstone [BdSBB17])**.** *Let $G(x,y)$ be a non-zero function in a quasianalytic class $\mathcal{Q}$, defined in a neighborhood $U \times V$ of $(a,b) \in \mathbb{R}^n \times \mathbb{R}$. There exists a quasianalytic class $\mathcal{Q}' \supseteqq \mathcal{Q}$ such that, if the equation*

$$G(x,y) = 0$$

admits a formal solution $y = H(x)$ in the point a (and $b = H(a)$), then there exists a solution $y = h(x) \in \mathcal{Q}'$ defined in a neighborhood of a, that is, $G(x,h(x)) \equiv 0$ and $Taylor_a(h) = H$. Furthermore, if $\mathcal{Q}$ is the class *of locally C^∞-definable functions over an o-minimal polynomially bounded structure, then $\mathcal{Q}' = \mathcal{Q}$.*

In the case that $G(x, y)$ is a monic polynomial in y with quasianalytic coefficients, we can prove a stronger version of the above result:

Theorem 5.2.2 (Belotto da Silva, Biborski, Bierstone [BdSBB17])**.** *Let $\mathcal{Q}$ denote a quasianalytic class. Let U denote a (connected) neighborhood of the origin in $\mathbb{R}^n$, with coordinates $x = (x_1, \ldots, x_n)$, and let*

$$G(x, y) = y^d + a_1(x)y^{d-1} + \cdots + a_d(x), \tag{5.1}$$

where the coefficients $a_i \in \mathcal{Q}(U)$. Let

$$G(x, y) = \prod_{j=1}^{k} \left(y^{d_j} + B_{j1}(x)y^{d_j-1} + \cdots + B_{j,d_j}(x)\right)$$

denote the irreducible factorization of $G(x, y)$ as an element of $\mathbb{R}[\![x]\!][y]$. Then there is a (perhaps larger) quasianalytic class $\mathcal{Q}' \supseteq \mathcal{Q}$ and a neighbourhood V of 0 in U, such that each B_{ji} is the formal Taylor expansion $\hat{b}_{ji,0}$ at 0 of an element $b_{ji} \in \mathcal{Q}'(V)$, and

$$G(x, y) = \prod_{j=1}^{k} \left(y^{d_j} + b_{j1}(x)y^{d_j-1} + \cdots + b_{j,d_j}(x)\right)$$

in $\mathcal{Q}'(V)[y]$. Furthermore, if $\mathcal{Q}$ is the class of locally C^∞-definable functions over an o-minimal polynomially bounded structure, then $\mathcal{Q}' = \mathcal{Q}$.

Note that Theorem 5.2.1 does not evidently reduce to the case of a monic polynomial equation because of the lack of a Weierstrass preparation theorem in quasianalytic classes.

Theorems 5.2.1, 5.2.2 and other related results are proved using techniques of *quasianalytic continuation*, *resolution of singularities* and *control on the estimates of Denjoy-Carleman classes.* See the sketch of the proof of Proposition 5.6.4 below for an idea of the philosophy involved. Note, furthermore, that previous results on this topic (e.g. [Now13, 2013]) demanded the existence of a formal solution *everywhere*, instead of over a unique point, because they lacked the technique of quasianalytic continuation.

5.3 Composition of quasianalytic maps [BdSBC18]

In a subsequent work in collaboration with Bierstone and Chow, we have extended our interest to composition of quasianalytic maps and the *image* of a quasianalytic map. More concretely, by improving the methods used in [BdSBB17] we proved the following results:

Theorem 5.3.1 (Belotto da Silva, Bierstone, Chow, [BdSBC18]). *Let $\mathcal{Q}_M$ denote a quasianalytic Denjoy-Carleman class, and let $\varphi : V \to W$ be a mapping of class $\mathcal{Q}_M$ between open sets $V \subset \mathbb{R}^m$ and $W \subset \mathbb{R}^n$, which is generically a submersion (i.e., generically of rank n). Let $f \in \mathcal{Q}_M(V)$. If $a \in V$ and $\hat{f}_a = G \circ \hat{\varphi}_a$, where G is a formal power series centred at $\varphi(a)$, then there is a relatively compact neighbourhood U of a in V and a function $g \in \mathcal{Q}_{M^{(p)}}(\varphi(\overline{U}))$, for some $p \in \mathbb{N}$, such that $f = g \circ \varphi$ on U.*

Theorem 5.3.2 (Belotto da Silva, Bierstone, Chow, [BdSBC18]). *Let $\mathcal{Q}_M$ denote a quasianalytic Denjoy-Carleman class, and let $\varphi : V \to W$ be a $\mathcal{Q}_M$-mapping of $\mathcal{Q}_M$-manifolds which is proper and generically submersive. Let $f \in \mathcal{Q}_M(V)$ and let $b \in \varphi(V)$. Suppose that $\hat{f}_a = G \circ \hat{\varphi}_a$, for all $a \in \varphi^{-1}(b)$, where G is a formal power series centred at b. Then, after perhaps shrinking W to an appropriate neighbourhood of b, there exists $g \in \mathcal{Q}_{M^{(p)}}(\varphi(V))$, for some p, such that $f = g \circ \varphi$.*

It is striking that, as a result of quasianalytic continuation, it is sufficient to assume that there is a formal solution at a single point in Theorems 5.3.1 and 5.3.2. In particular, Theorem 5.3.1 reduces to the classical statement in the real-analytic case (cf. [BM88, Lemma 7.8], [Mal77]), even though it seems unlikely that the ring of formal power series at a point is flat over the local ring of germs of functions of class $\mathcal{Q}_M$, in general. Note that in previous works of Chaumat and Chollet, [ChCh99] and [ChCh01], it was necessary to suppose the existence of a formal solution at every point.

5.4 Sharp estimates for blowing down functions in a Denjoy-Carleman class [BdSBK20]

We follow closely [BdSBK20, § 1]. Our ambition is to study if the loss of regularity in Theorems 5.3.1 and 5.3.2 is sharp, whenever the mapping φ is a blowings-up. More precisely, we can express the blowing up of the origin in the plane $\mathbb{R}^2$ using polar coordinates, as the mapping $\sigma : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\sigma(r, \theta) = (r \cos\theta,\, r \sin\theta)$$

(the universal covering of the standard blowing-up). If $F \in C^\infty(\mathbb{R}^2)$ is a function such that $F \circ \sigma \in C_M(\mathbb{R}^2)$, then F belongs to the shifted class $C_{M^{(2)}}(\mathbb{R}^2)$ (see definition in §3.6), by [BdSBB17, Lemma 3.4]. We show that this estimate is sharp.

Theorem 5.4.1 (Belotto da Silva, Bierstone, Kiro [BdSBK20])**.** *Let C_M be a Denjoy-Carleman class closed under differentiation, such that $C_{M^2} = C_{M^{(2)}}$. Then, for every Denjoy-Carleman class $C_N \subsetneq C_{M^{(2)}}$, there exists $F \in C_{M^{(2)}}(\mathbb{R}^2) \setminus C_N(\mathbb{R}^2)$ such that $F \circ \sigma \in C_M(\mathbb{R}^2)$.*

Moreover, under the hypothesis that the class C_M is quasianalytic, our techniques provide the following result.

Theorem 5.4.2 (Belotto da Silva, Bierstone, Kiro [BdSBK20])**.** *Let C_M be a quasianalytic Denjoy-Carleman class closed under differentiation. For every Denjoy-Carleman class $C_N \subsetneq C_{M^{(2)}}$ such that $\lim(N_k/M_{2k})^{1/k} = 0$, there exists $F \in C_{M^{(2)}}(\mathbb{R}^2) \setminus C_N(\mathbb{R}^2)$ such that $F \circ \sigma \in C_M(\mathbb{R}^2)$.*

In particular, if C_M properly contains the analytic functions, then there exists $F \in C_{M^{(2)}}(\mathbb{R}^2) \setminus C_M(\mathbb{R}^2)$ such that $F \circ \sigma \in C_M(\mathbb{R}^2)$.

Solutions of problems on Denjoy-Carleman classes C_M using resolution of singularities, in general lead to *loss of regularity*, (c.f. Theorems 5.2.1, 5.2.2, 5.3.1 and 5.3.2). Theorems 5.4.1 and 5.4.2 show that loss of regularity is an essentially unavoidable limitation of the technique of resolution of singularities. It seems important, therefore, to understand whether loss of regularity is a limitation only of the technique, or is intrinsic to geometric questions on Denjoy-Carleman classes. For example, if Theorem 5.2.1 is sharp, and one can not avoid loss of regularity when dividing, this would imply that local rings of quasianalytic Denjoy-Carleman functions are not Noetherian.

5.5 Applications of the monomialization of quasianalytic morphisms

As we presented in §4.4, in collaboration with Bierstone, we have combined our previous composition results with techniques of *resolution of singularities tangent to log-differentials* (see §4.6) in order to obtain a *monomialization of quasianalytic morphisms*. Monomialization is a version of resolution of singularities for morphisms. More precisely, we prove that a mapping in a real quasianalytic class can be transformed by sequences of simple changes of the source and target (sequences of local blowings-up and local power substitutions) to a mapping whose components are monomials with respect to suitable local coordinate systems. We refer to §4.4 for details, and we reproduce the statement of Theorem 4.4.4:

Theorem 5.5.1 (Belotto da Silva, Bierstone [BdSB19]). *Suppose that $\mathcal{Q}$ is a real quasianalytic class. Let $\Phi : (M, D) \to (N, E)$ be a $\mathcal{Q}$-map and fix $\mathfrak{a} \in M$. Then there is an open neighbourhood V of $\mathfrak{a}$ and a finite number of commutative diagrams*

$$\begin{array}{ccc} (V_\lambda, D_\lambda) & \xrightarrow{\sigma_\lambda} & (M, D) \\ \downarrow{\scriptstyle \Phi_\lambda} & & \downarrow{\scriptstyle \Phi} \\ (W_\lambda, E_\lambda) & \xrightarrow{\tau_\lambda} & (N, E) \end{array},$$

where

1. *each σ_λ and τ_λ is a composite of finitely many smooth local blowings-up and local power substitutions, compatible with D and E, respectively;*

2. *the families of morphisms $\{\sigma_\lambda\}$ and $\{\tau_\lambda\}$ cover V and W, respectively, and there are compact subsets $K_\lambda \subset V_\lambda$, $L_\lambda \subset W_\lambda$, such that $\bigcup \sigma_\lambda(K_\lambda)$ and $\bigcup \sigma_\lambda(L_\lambda)$ are (compact) neighbourhoods of a and b, respectively;*

3. *each Φ_λ is a monomial morphism (see Definition 4.4.1).*

In what follows, we discuss applications of the monomialization Theorem to quasianalytic geometry and model theory which can be found in [BdSB19, Section 1.3].

For a real quasianalytic class $\mathcal{Q}$, we can define *sub-quasianalytic* functions and sub-quasianalytic sets in a straightforward way generalizing sub-analytic. More precisely, we say that a set $X \subset \mathbb{R}^n$ is sub-quasianalytic, if there exists a projection $\pi : \mathbb{R}^{n+m} \to \mathbb{R}^n$ and $\mathcal{Q}$-set $Y \subset \mathbb{R}^{n+m}$ such that $\pi(Y) = X$ and $\pi|_Y$ is a proper mapping. As a direct consequence of our monomialization Theorem, we obtain:

Theorem 5.5.2 (Characterization of a sub-quasianalytic function, [BdSB19]). *Let $\mathcal{Q}$ denote a real quasianalytic class, and let $f : N \to \mathbb{R}$ be a continuous function. Then f is sub-quasianalytic if and only if there is a countable semiproper covering $\{\tau_\lambda : W_\lambda \to N\}$ of N of class $\mathcal{Q}$, such that*

1. *each morphism* τ_λ *is a composite of finitely many smooth local blowings-up and power substitutions;*

2. *each function* $f_\lambda := f \circ \tau_\lambda$ *is quasianalytic of class* $\mathcal{Q}$.

Theorem 5.5.2 in the real analytic case was proved in [BM90], c.f. [DvdD88, (4.6)]. Note that, even in the real-analytic case, the analogous assertion using local blowings-up alone is not true. (A real-valued function f can be transformed to analytic by sequences of local blowings-up of the source if and only if f is subanalytic and *arc-analytic*; i.e., analytic on every real-analytic arc [BM90]). Theorem 5.5.2, in general, has been proved by Rolin and Servi using model-theoretic techniques [RS15, Thm. 4.2]. Rolin and Servi show that quantifier elimination in the structure defined by restricted functions of class $\mathcal{Q}$ (together with reciprocal and nth roots) is a direct consequence of Theorem 5.5.2 (see [RS15, Sect. 4]).

Theorems 5.5.3, 5.5.4 following, which are obtained as Corollaries of our main monomialization result, are strong versions for real quasianalytic classes of Hironaka's rectilinearization theorem for subanalytic sets, c.f. [Hi73, Thm. 7.1], [BM88, Thm. 0.2], [DvdD88, page 82] and [RS15, Sect. 4].

Theorem 5.5.3 (Rectilinearization I, [BdSB19])**.** *Let* N *denote a manifold of real quasianalytic class* $\mathcal{Q}$*, and let* X *denote a sub-quasianalytic subset of* N*. Assume that* N *has pure dimension* n*. Then there is a countable semiproper covering* $\{\tau_\lambda : W_\lambda \to N\}$ *of* N *of class* $\mathcal{Q}$*, such that, for each* λ*,*

1. $W_\lambda \cong \mathbb{R}^n$ *and* τ_λ *is a composite of finitely many smooth local blowings-up and power substitutions compatible with the exceptional divisors;*

2. $\tau_\lambda^{-1}(X)$ *is a union of orthants of* $\mathbb{R}^n_\lambda$.

Theorem 5.5.4 (Rectilinearization II, [BdSB19])**.** *Let* N *denote a manifold of real quasianalytic class* $\mathcal{Q}$*, with SNC divisor* E*. Assume that* N *has pure dimension* n*. Let* X *denote a sub-quasianalytic subset of* N*. Then there is a countable semiproper covering* $\{\tau_\mu : W_\mu \to N\}$ *of* N *of class* $\mathcal{Q}$*, such that,*

1. *for each* μ*,* W_μ *is a copy* $\mathbb{R}^n_\mu$ *of* $\mathbb{R}^n$ *and* τ_μ *is a composite of finitely many smooth local blowings-up and power substitutions compatible with the divisors;*

2. $X \backslash E = \cup_\mu \tau_\mu(\mathbb{R}^{k(\mu)}_\mu \backslash E_\mu)$*, where, for each* μ*,* $\mathbb{R}^{k(\mu)}_\mu$ *is a coordinate subspace of* $\mathbb{R}^n_\mu$ *(of dimension* $k(\mu)$*) transverse to the divisor* $E_\mu \subset \mathbb{R}^n_\mu$*,*

and τ_μ *restricts to an embedding on each connected component of* $\mathbb{R}_\mu^{k(\mu)} \backslash E_\mu$.

5.6 Quasianalytic continuation

The quasianalyticity axiom 3.6.1(3) provides a generalization of the classical property of analytic continuation. We recall that a common way to define functions in analysis proceeds by first specifying the function on a small open set, and extending it (whenever possible) by analytic continuation. In practice, in order to guarantee the existence of such a continuation, it is useful to first establish some functional equation on the small open set, which can be extended (c.f. the Riemann zeta function; the Gamma function).

In our context, we use quasianalyticity to show that, if the formal Taylor expansion $\hat{f}_{\mathfrak{a}}$ of a quasianalytic function f at a given point $\mathfrak{a}$ is the composite $H \circ \hat{\varphi}_{\mathfrak{a}}$ of a formal power series H with the formal expansion of a suitable quasianalytic mapping φ, then this formal composition property extends to a neighbourhood of $\mathfrak{a}$. This is a powerful tool when combined with axioms 3.6.1(1,2), which allow one to solve "reduced" quasianalytic equations, and resolution of singularities, which allow one to "reduce" quasianalytic equations. Proposition 5.6.4 below provides a concrete example of this philosophy. For now, let us start by providing the precise statement of quasianalytic continuation:

Theorem 5.6.1 (Quasianalytic continuation I, [BdSBB17, BdSBC18])**.** *Let* $\mathcal{Q}$ *denote a quasianalytic class (see Definition 3.6.1). Let* $\varphi : V \to W$ *denote a* $\mathcal{Q}$*-mapping, where* V *is a* $\mathcal{Q}$*-manifold and* W *is an open neighbourhood of* $0 \in \mathbb{R}^n$. *Let* $f \in \mathcal{Q}(V)$ *and let* H *denote a formal power series at* $0 \in \mathbb{R}^n$. *Then:*

1. *The set* $\{\mathfrak{a} \in \varphi^{-1}(0) : \hat{f}_{\mathfrak{a}} = H \circ \hat{\varphi}_{\mathfrak{a}}\}$ *is open and closed in* $\varphi^{-1}(0)$.

2. *Suppose that* φ *is proper and generically of rank* $n = dim(W)$. *Assume* $\hat{f}_{\mathfrak{a}} = H \circ \hat{\varphi}_{\mathfrak{a}}$, *for all* $\mathfrak{a} \in \varphi^{-1}(0)$. *Then, after perhaps shrinking* W, f *is* formally composite *with* φ*; i.e., for all* $\mathfrak{b} \in W$, *there exists a power series* $H_{\mathfrak{b}}$ *centred at* $\mathfrak{b}$ *such that* $\hat{f}_{\mathfrak{a}} = H_{\mathfrak{b}} \circ \hat{\varphi}^*_{\mathfrak{a}}$, *for all* $\mathfrak{a} \in \varphi^{-1}(\mathfrak{b})$.

The above result is particularly useful when used in combination with Glaeser Proper Mapping Theorem, which states that a map is a C^∞-composition (with a suitable map φ) if, and only if, it is an everywhere formal composition:

Theorem 5.6.2 (Glaeser Proper Mapping Theorem [Gl63])**.** *Let $\varphi : V \to W$ denote a proper analytic mapping, generically a submersion, and let $f \in C^\infty(V)$. Then, there exists a function $h \in C^\infty(\varphi(V))$ such that $f = h \circ \varphi$ if, and only if, at every point $\mathfrak{a} \in \varphi(V)$, there exists a formal power series $H_\mathfrak{a}$ such that, $\hat{f}_\mathfrak{b} = H_\mathfrak{a} \circ \widehat{\varphi}_\mathfrak{b}$ for every point $\mathfrak{b} \in \varphi^{-1}(\mathfrak{a})$.*

The above Theorem is also valid for proper (or even semi-proper) quasianalytic mappings $\varphi : V \to W$, as was proven by Nowak [Now11]. For most of our intended applications, nevertheless, the original Theorem of Glaeser is enough because blowings-up and power substitutions are analytic maps (at least locally, with respect to suitable coordinate systems) and, whenever φ is a resolution of singularities, we can use Glaeser proper mapping Theorem one blowing-up at a time. In either case, it follows that:

Corollary 5.6.3 (Quasianalytic continuation II,[BdSBB17, BdSBC18])**.** *Let $\mathcal{Q}$ denote a quasianalytic class (see Definition 3.6.1). Let $\varphi : V \to W$ denote a proper $\mathcal{Q}$-mapping generically of rank $n = dim(W)$ such that $\varphi(V) = W$, where V is a $\mathcal{Q}$-manifold and W is an open neighbourhood of $0 \in \mathbb{R}^n$. Let $f \in \mathcal{Q}(V)$ and let H denote a formal power series at $0 \in \mathbb{R}^n$, and assume that $\hat{f}_\mathfrak{a} = H \circ \hat{\varphi}_\mathfrak{a}$, for all $\mathfrak{a} \in \varphi^{-1}(0)$. Then, after perhaps shrinking W, f is C^∞-*composite *with φ; i.e., there exists a function $h \in C^\infty(W)$ such that $f = h \circ \varphi$.*

Before sketching the proof of Theorem 5.6.1, let us illustrate the interest of quasianalytic continuation by proving that principal ideals are closed whenever $\mathcal{Q}$ is a class of C^∞-definable functions. Note that this is a particular case of Theorem 5.2.1.

Proposition 5.6.4 (Belotto da Silva, Bierstone, Biborski [BdSBB17])**.** *Let $\mathcal{Q}$ be a quasianalytic class and let $g \in \mathcal{Q}(W)$, where W is a neighbourhood of 0 in $\mathbb{R}^n$. Then there is a quasianalytic class $\mathcal{Q}' \supset \mathcal{Q}$ such that, given $f \in \mathcal{Q}(W)$ and a formal power series $H \in \mathbb{R}[[x]]$ such that $\widehat{f}_0 = H \cdot \widehat{g}$, apart from shrinking W, there exists $h \in \mathcal{Q}'(W)$ such that $f = h \cdot g$. Moreover, if $\mathcal{Q}$ is the class of C^∞-definable functions, than $\mathcal{Q}' = \mathcal{Q}$.*

Sketch of the proof: If $n = 1$ the result is straightforward[1]. Indeed, since $g \in \mathcal{Q}$, either $g \equiv 0$, or $f \equiv 0$ (and, in both cases, the result is evident), or $\widehat{g}_0 = x^\alpha \xi(x)$ and $\widehat{f}_0 = x^\beta \eta(x)$ with $\xi(0) \neq 0$ and $\eta(0) \neq 0$. It follows from

[1]It is easy to verify that local quasianalytic rings are Noetherian when $n = 1$, immediately implying the result. In what follows, we provide an elementary argument which is useful later on.

hypothesis of a formal solution that $\beta \leq \alpha$, and by the quasianalytic axiom 3.6.1(1) that $h(x) = x^{\beta-\alpha}\eta(x)/\xi(x)$ belongs to $\mathcal{Q}$.

If $n > 1$, nevertheless, the situation is more complex essentially because the formal expansions of f and g are not monomial, so we can not divide coordinate by coordinate via axiom 3.6.1(1). This is always possible, nevertheless, after a *resolution of singularities* of g. More precisely, let $\varphi : V \to W \subset U$ be a resolution of singularities of $g(x)$, that is, for every $\mathfrak{b} \in W$ and $\mathfrak{a} \in \varphi^{-1}(\mathfrak{b})$, we have that $T_{\mathfrak{b}}(g \circ \varphi) = x^{\alpha}\xi(x)$, where $x^{\alpha} = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $\xi(\mathfrak{a}) \neq 0$. It follows from the hypothesis, that $T_{\mathfrak{a}}(g \circ \varphi)$ formally divides $T_{\mathfrak{a}}(f \circ \varphi)$, which means that x^{α} divides $T_{\mathfrak{a}}(f \circ \varphi)$. It follows from the quasianalytic axiom 3.6.1(1), therefore, that $h_{\mathfrak{a}} = (f \circ \varphi)/(g \circ \varphi)$ belongs to $\mathcal{Q}(V_{\mathfrak{a}})$, for some open neighbourhood $V_{\mathfrak{a}}$ of $\mathfrak{a}$, and that $T_{\mathfrak{a}}(h_{\mathfrak{a}}) = H \circ \widehat{\varphi}_{\mathfrak{a}}$. By the quasianalytic axiom 3.6.1(3), apart from shrinking V, we can patch all of the locally defined functions $\xi_{\mathfrak{a}}$ into a globally defined function $h_V \in \mathcal{Q}(V)$ (strictly saying, it is necessary to use Lemma 5.6.6 below, which involves a topological argument, to prove that the gluing is well-defined).

Now, by quasianalytic continuation 5.6.3, there exists a C^{∞}-function $h \in C^{\infty}(W)$ such that $h \circ \varphi = h_V$ and, by construction, $f = g \cdot h$. Let $\mathcal{Q}'$ be the class of locally C^{∞}-definable functions on the o-minimal structure generated by restricted $\mathcal{Q}$-functions[2] (so $\mathcal{Q}' = \mathcal{Q}$ if $\mathcal{Q}$ is already a class of locally C^{∞} definable functions). Since $h \circ \varphi \in \mathcal{Q}'$, and φ is a $\mathcal{Q}'$-definable map, it easily follows that $h \in \mathcal{Q}'(W)$ as we wanted to prove. □

Sketch of the proof of Theorem 5.6.1. Part (1) of the Theorem is formalized in [BdSBB17, Proposition 4.6], and follows from elementary considerations about composition and quasianalyticity. We omit its details, and we concentrate in part (2). Its proof can be reduced, via standard arguments of topology and resolution of singularities, to the following Lemma (whose proof we sketch in the end of this section):

Lemma 5.6.5. *Let U, W denote open neighbourhoods of the origin in $\mathbb{R}^n$, with coordinate systems $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, respectively. (Assume U is chosen so that every coordinate hyperplane $(x_i = 0)$ is connected). Let $\varphi : U \to W$ denote a $\mathcal{Q}$-mapping such that the Jacobian determinant* $\det(\partial\varphi/\partial x)$ *is a monomial times an invertible factor in $\mathcal{Q}(U)$. Let $f \in \mathcal{Q}(U)$ and let $H \in \mathbb{R}[[x]]$ be a formal power series centred at $0 \in W$, such that $\hat{f}_0 = H \circ \widehat{\varphi}_0$. Then, for all $\beta \in \mathbb{N}^n$, there exists $f_\beta \in \mathcal{Q}(U)$ such that $f_0 = f$ and*

[2] It is possible to provide a tighter loss of regularity for Denjoy-Carleman classes, see [BdSBB17, §3] and [BdSBC18, §5].

1. *for all* $\mathfrak{a} \in U$, $\hat{f}_{\mathfrak{a}} = H_{\mathfrak{a}} \circ \hat{\sigma}_{\mathfrak{a}}$, *where* $H_{\mathfrak{a}}$ *is a formal power series centred at* $\varphi(\mathfrak{a})$ *given by*
$$H_{\mathfrak{a}} := \sum_{\beta \in \mathbb{N}^n} \frac{f_\beta(\mathfrak{a})}{\beta!} y^\beta; \tag{5.2}$$

2. *each* f_β, $\beta \in \mathbb{N}^n$, *and therefore also* $H_{\mathfrak{a}}$ *(as a function of* $\mathfrak{a}$*) is constant on connected components of the fibres of* φ.

3. *if for every* $\mathfrak{a} \in \varphi^{-1}(0)$ *we have that* $\hat{f}_{\mathfrak{a}} = H \circ \varphi_{\mathfrak{a}}$, *then each* f_β, $\beta \in \mathbb{N}^n$, *and therefore also* $H_{\mathfrak{a}}$, *is constant along the fibres of* φ. *In other words, apart from shrinking* W, *for every* $\mathfrak{b} \in W$, *there exists* $H_{\mathfrak{b}}$ *such that* $H_{\mathfrak{b}} = H_{\mathfrak{a}}$ *for every* $\mathfrak{a} \in \varphi^{-1}(\mathfrak{b})$.

4. *Under the same hypothesis as in* (3), *if* H *is independent of some variable* y_j, *then* $H_{\mathfrak{b}}$ *is independent of* y_j, *for all* $\mathfrak{b} \in V$.

Indeed, if we assume that $\dim(V) = \dim(W) = n$, then apart from using resolution of singularities, we can always suppose that the Jacobian determinant $\det(\partial\varphi/\partial x)$ is everywhere locally given by a monomial times an invertible factor (such as in the hypothesis of the previous Lemma). In this case, Theorem 5.6.1 follows from applying Lemma 5.6.5 at every point $\mathfrak{a} \in \varphi^{-1}(0)$, and "glueing" the local solutions f_β along the fibre $\varphi^{-1}(0)$ (which is compact, from the assumption that φ is proper). Indeed, this glueing is possible due to the following Lemma (whose proof follows from a topological argument which we omit, but can be find in [BdSBB17, Lemma 4.4]):

Lemma 5.6.6. *Let* $\varphi : V \to W$ *denote a proper* $\mathcal{Q}$*-mapping, where* V *is a* $\mathcal{Q}$*-manifold of dimension* n, *and* W *is an open neighbourhood of the origin in* $\mathbb{R}^n$. *Then, given any open covering* $\{U\}$ *of* $\varphi^{-1}(0)$, *there is a neighbourhood* $\widetilde{W}$ *of* 0 *in* W *with the following properties:*

1. $\varphi^{-1}(\widetilde{W}) \subset \bigcup U$.

2. *Let* H *be a power series centred at* $0 \in W$, *and suppose there exists* $f_U \in \mathcal{Q}(U)$, *for each* U, *such that* $\hat{f}_{U,\mathfrak{a}} = H \circ \widehat{\varphi}_{\mathfrak{a}}$, *for all* $\mathfrak{a} \in \varphi^{-1}(0) \cap U$. *Then there exists* $f \in \mathcal{Q}(\varphi^{-1}(\widetilde{W}))$ *such that* $\hat{f}_{\mathfrak{a}} = H \circ \widehat{\varphi}_{\mathfrak{a}}$, *for all* $\mathfrak{a} \in \varphi^{-1}(0)$.

The general case, that is, when $\dim(V) > \dim(W) = n$ was proved in our later work [BdSBC18] and can be reduced, essentially, to the first case by "adding locally defined coordinates". More precisely, write $\varphi =$

$(\varphi_1,\dots,\varphi_n)$. Every point of $\varphi^{-1}(0)$ has a neighbourhood U in V with a coordinate system $(x_1,\dots,x_m)$ in which the Jacobian submatrix

$$\frac{\partial(\varphi_1,\dots,\varphi_n)}{\partial(x_1,\dots,x_n)}$$

is generically of rank n. Fix such U and define $\psi_U : U \to \mathbb{R}^m$ by

$$\psi_U(x_1,\dots,x_m) = (\varphi(x), x_{n+1},\dots,x_m). \tag{5.3}$$

Then, for all $\mathfrak{a} \in \varphi^{-1}(0)$,

$$\hat{f}_{\mathfrak{a}} = H \circ \hat{\varphi}_{\mathfrak{a}} = H_U \circ \hat{\psi}_{U,\mathfrak{a}},$$

where $H_U(y_1,\dots,y_m) = H(y_1,\dots,y_n)$; in particular, the formal power series $H_U(y)$ is independent of $y_{n+1},\dots,y_m$. We now can argue locally via Lemma 5.6.5 just as before, and we glue these solutions. We note that condition (4) of Lemma 5.6.5 plays a crucial role, in order to obtain power series which are independent of "artificial" variables $(y_{n+1},\dots,y_m)$. The technical details of this glueing procedure are given in [BdSBC18, page 8], and we omit it in here. We finish this section by sketching the proof of Lemma 5.6.5:

Sketch of the proof of Lemma 5.6.5. The proof of the Lemma follows from combining the proofs of [BdSBB17, Theorem 4.1] and [BdSBC18, Lemma 3.1]. Parts (2-3) follow from a geometrical argument which is essentially from Nowak [Now13], and which we adapt to our setting in [BdSBB17, Lemma 4.2]. We turn to the proof of parts (1) and (4).

The following proof has been almost entirely copied from [BdSBB17, Theorem 4.1], besides changes of notation and an extra argument in order to prove (4) simultaneously. Write $\varphi = (\varphi_1,\dots,\varphi_n)$ with respect to the coordinates of V. As formal expansions at $0 \in U$,

$$\sum_{j=1}^{n} \left(\frac{\partial H}{\partial y_j} \circ \varphi\right) \cdot \frac{\partial \varphi_j}{\partial x_i} = \frac{\partial f}{\partial x_i}, \quad i = 1,\dots,n,$$

so that

$$\det\left(\frac{\partial \varphi}{\partial x}\right) \cdot \left(\frac{\partial H}{\partial y_j} \circ \varphi\right) = \left(\frac{\partial \varphi}{\partial x}\right)^{*} \left(\frac{\partial f}{\partial x_i}\right),$$

where $(\partial f/\partial x_i)$ denotes the column vector with components $\partial f/\partial x_i$, and $(\partial\varphi/\partial x)^*$ is the adjugate matrix of $\partial\varphi/\partial x$.

By axioms 3.6.1(1), (3), for each $j = 1, \ldots, n$, there is a quasianalytic function $f_{(j)} \in \mathcal{Q}(U)$ such that

$$\hat{f}_{(j),0} = \frac{\partial H}{\partial y_j} \circ \widehat{\varphi}_0$$

where, if the formal power series H is independent of y_j, then $\hat{f}_{(j),0} \equiv 0$, and

$$\det\left(\frac{\partial \varphi}{\partial x}\right) \cdot \left(f_{(j)}\right) = \left(\frac{\partial \varphi}{\partial x}\right)^* \left(\frac{\partial f}{\partial x_i}\right)$$

in $\mathcal{Q}(U)$. It follows by induction on the order of differentiation that, for each $\beta \in \mathbb{N}^n$, there is a quasianalytic function $f_\beta \in \mathcal{Q}(U)$ such that

$$\hat{f}_{\beta,0} = \frac{\partial^{|\beta|} H}{\partial y^\beta} \circ \hat{\varphi}_0$$

such that, if H is independent of y_j and $\beta_j \neq 0$, then $\hat{f}_{\beta,0} \equiv 0$, and

$$\det\left(\frac{\partial \varphi}{\partial x}\right) \cdot \left(f_{\beta+(j)}\right) = \left(\frac{\partial \varphi}{\partial x}\right)^* \left(\frac{\partial f_\beta}{\partial x_i}\right).$$

Therefore, for all $\mathfrak{a} \in U$, $\hat{f}_{\mathfrak{a}} = H_{\mathfrak{a}} \circ \hat{\varphi}_{\mathfrak{a}}$, where $H_{\mathfrak{a}}$ is the formal power series centred at $\varphi(\mathfrak{a}) \in V$ given by (5.2). Likewise, for all $\beta \in \mathbb{N}^n$ and $\mathfrak{a} \in U$,

$$\hat{f}_{\beta,\mathfrak{a}} = \frac{\partial^\beta H_{\mathfrak{a}}}{\partial y^\beta} \circ \hat{\varphi}_{\mathfrak{a}} \qquad (5.4)$$

We conclude easily. □

Chapter 6

Sard Conjecture in Sub-Riemannian geometry

6.1 Preliminaries in Sub-Riemannian geometry and the Sard Conjecture

Let M be a smooth connected Riemannian manifold of dimension $n \geq 3$ and Δ a *nonholonomic distribution* of rank $k < n$ on M, that is, a smooth subbundle of TM of dimension k generated locally by k smooth vector fields $X^1, \ldots, X^k$ satisfying the Hörmander condition. More precisely:

Definition 6.1.1 (Non-holonomic distribution). *Consider the sequence of (singular) distributions:*

$$\Delta_0 = \Delta_0, \quad \Delta_{i+1} = \Delta_i + [\Delta_i, \Delta_i], \quad \Delta_\infty = \bigcup_{i \in \mathbb{N}} \Delta_i,$$

where $[\Delta, \Delta]$ *is the (possibly singular) distribution given by*

$$[\Delta, \Delta]_x := \Big\{ [X, Y](x) \,|\, X, Y \text{ smooth local sections of } \Delta \Big\}.$$

The distribution Δ *is non-holonomic if, and only if,*

$$Lie(\Delta) := \Delta_\infty = TM$$

In particular, note that $k > 1$; indeed, if $k = 1$, then $Lie(\Delta) = \Delta$ and the distribution can not be non-holonomic, unless $k = 1$.

Roughly, sub-Riemannian geometry studies the "geometry" of the paths in M which are almost everywhere tangent to Δ. More precisely, consider:

Definition 6.1.2 (Horizontal path)**.** *An horizontal curve (with respect to* Δ*) is an absolutely continuous*[1] *curve* $\gamma : [0,1] \to M$ *such that:*

$$\dot{\gamma}(t) \in \Delta_{\gamma(t)}, \qquad \textit{for almost every } t \in [0,1]$$

(where "almost every" is in the sense of measure theory, that is, outside of a set of Lebesgue measure zero).

By the Chow-Rashevsky Theorem, M is horizontally path-connected with respect to Δ. In other words, for every pair of points $x, y \in M$ there is a horizontal path $\gamma : [0,1] \to M$ connecting them. The sub-Riemannian distance between two points x and y in M (denoted by $d_{SR}(x,y)$) is the infimum of the length of every horizontal path between x and y. Note that every horizontal path has finite length (every absolutely continuous curve defined in a compact set has finite length) and, therefore, the SR-distance is always finite.

In order to study horizontal paths, we consider a "parametrization" of all horizontal curves. More precisely, suppose that there exist globally defined smooth vector-fields $\{X^1, \ldots, X^k\}$ which generate Δ (note that this condition is always verified over sufficiently small open sets; we refer the reader to [Ag14, BdSR18, Mo02, Ri14] for the general presentation). For every $x \in M$, the set of *controls* $u = (u_1, \ldots, u_k) \in L^2([0,1], \mathbb{R}^k)$ for which the solution $\mathbf{x}(\cdot) = \mathbf{x}(\cdot\,; x, u)$ to the Cauchy problem

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{k} u_i(t) X^i(\mathbf{x}(t)) \quad \text{for a.e. } t \in [0,1] \quad \text{and} \quad \mathbf{x}(0) = x,$$

exists over $[0,1]$ is a nonempty open set $\mathcal{U}^x \subset L^2([0,1], \mathbb{R}^k)$. By construction, any solution $\mathbf{x}(\cdot; x, u) : [0,1] \to M$ with $u \in \mathcal{U}^x$ is a horizontal path in M. Moreover, by definition, any horizontal path $\gamma : [0,1] \to M$ is equal, apart from re-parametrization, to a solution $\mathbf{x}(\cdot\,; x, u)$ for some $u \in \mathcal{U}^x$. Given a point $x \in M$, the *End-Point Mapping* from x is defined as

$$\begin{array}{rccl} \mathrm{E}^x : & \mathcal{U}^x & \longrightarrow & M \\ & u & \longmapsto & \mathbf{x}(1; x, u), \end{array}$$

and it is of class C^∞ on $\mathcal{U}^x$ equipped with the L^2-topology.

[1]We recall that a function $f : [0,1] \to \mathbb{R}$ is absolutely continuous if its derivative f' is defined almost everywhere, Lebesgue integrable, and $f(x) = f(0) + \int_0^x f'(t)\,dt$ for all $x \in [0,1]$. Note that the set of absolutely continuous functions over $[0,1]$ form a Banach space; this fact is not explicitly used at any point of this memoir, but is an important point for the general theory.

Definition 6.1.3 (Singular horizontal path)**.** *A control u in an open subset $\mathcal{U}^x \subset L^2([0,1], \mathbb{R}^k)$ is called singular (with respect to E^x) if it is a critical point of the map E^x. In this case, the associated horizontal path $\gamma_u \in \Omega^x_\Delta$ is called* singular.

Remark 6.1.4 (Horizontal geodesics)**.** *In order to illustrate the importance of the above concepts, consider the problem of finding horizontal-geodesics between x and $y \in M$. Suppose that the global generators $\{X^1, \dots, X^k\}$ form an orthonormal frame, and recall that every horizontal path is related to a control $u \in \mathcal{U}^x$. Note that:*

$$\|u\|_{L^2} = \int_0^1 \left(\sum_{i=1}^k u_i^2(t) \right)^{1/2} dt = \int_0^1 \|\dot{\mathbf{x}}(x,t,u)\| dt = \mathit{length}(\mathbf{x}(x,t,u)).$$

Therefore, finding an horizontal geodesic between x and y is equivalent to:

$$\textit{Minimize } \|u\|^2_{L^2} \textit{ where } u \in \mathcal{U}^x \textit{ respects the constraint } E^x(u) = y.$$

Recall that Lagrange multiplier is a typical method to solve this kind of problem. Such a method does not distinguish the minima, nevertheless, from critical points of the constraint condition ($E^x(u) = y$), which are precisely the singular controls.

Definition 6.1.5 (Critical values of the end-point mapping)**.** *For every $x \in M$, we denote by $\mathcal{S}^x_\Delta$ the set of singular horizontal paths starting at x and we set*

$$\mathcal{X}^x_\Delta := \left\{ \gamma(1) \,|\, \gamma \in \mathcal{S}^x_\Delta \right\} \subset M.$$

By construction, the set $\mathcal{X}^x_\Delta$ coincides with the set of critical values of the smooth mapping E^x. In analogy with the classical Sard Theorem in finite dimension, the Sard conjecture in sub-Riemannian geometry (first formulated in an unpublished work of Montgomory and Zhitomirskii [Mo02, page 140]) asserts the following:

Sard Conjecture. *For every $x \in M$, the set $\mathcal{X}^x_\Delta$ has Lebesgue measure zero in M.*

The Sard Conjecture cannot be obtained as a straightforward consequence of a general Sard Theorem in infinite dimension, as the latter fails to exist [BaMo01]. According to Montgomery, "A positive answer [to the Sard Conjecture] would lead to a fundamental progress in understanding the structure of geodesics" [Mo02, 140 pp]. This conjecture, nevertheless, remains largely open, except for particular cases in dimension $n \geq 4$ (see [DMOPV16, Mo02, Ri17]) and for the 3-dimensional case where a stronger conjecture is expected.

6.2 Sard Conjecture in dimension 3.

If $\dim M = 3$ (in which case $1 < m < 3$ means that $m = 2$), then the Sard Conjecture holds true by a simple argument, and a stronger Conjecture is expected. Indeed, consider the *Martinet surface* Σ associated to Δ, which is defined as

$$\Sigma := \Big\{ x \in M \mid \Delta_x + [\Delta, \Delta]_x \neq T_x M \Big\}.$$

Remark 6.2.1 (Local Model)**.** *Locally, we can always suppose that M coincides with a connected open subset $\mathcal{V} \subset \mathbb{R}^3$, and that Δ is everywhere generated by global smooth sections. More precisely, we can choose one of the following equivalent formulations:*

(i) Δ is a nonholonomic distribution generated by a smooth 1-form δ (that is, a section in $\Omega^1(M)$) and

$$\delta \wedge d\delta = h \cdot \omega_M, \tag{6.1}$$

where h is a smooth function defined in M whose zero locus defines the Martinet surface (that is, $\Sigma = \{p \in M \mid h(p) = 0\}$) and ω_M is a local volume form.

(ii) Δ is generated by two global smooth vector fields X^1 and X^2 which satisfy the Hörmander condition, and $[\Delta, \Delta]$ is generated by X^1, X^2, and $[X^1, X^2]$. Also, up to using the Flow-box Theorem and taking a linear combination of X^1 and X^2, we can suppose that

$$X^1 = \partial_{x_1}, \quad X^2 = \partial_{x_2} + A(x)\,\partial_{x_3}, \quad [X^1, X^2] = A_1(x)\,\partial_{x_3},$$

where (x_1, x_2, x_3) is a coordinate system on M, and $A_1(x) := \partial_{x_1} A(x)$. In this case, the zero locus of $A_1(x)$ defines the Martinet surface (that is, $\Sigma = \{p \in M \mid A_1(p) = 0\}$).

Remark 6.2.2 (The Hausdorff dimension of the Martinet surface Σ)**.** *We call the set Σ a "surface" because it has Hausdorff dimension at most two. Indeed, by non-holonomicity, the Martinet surface is everywhere locally generated by a smooth function h with non-zero Taylor expansion, c.f. Remark 6.2.1. The results now easily follows from Malgrange Preparation Theorem.*

Remark 6.2.3 (Analytic Martinet surfaces)**.** *If M is a real-analytic manifold and Δ is a coherent analytic distribution, then the Martinet surface is a real-analytic set of dimension at most two. Furthermore, the Martinet surface Σ admits a structure of a reduced and coherent analytic space $(\Sigma, \mathcal{O}_\Sigma)$ according to [BdSR18, Appendix C]. In particular, if follows from Remark 3.1.1 that Sing$((\Sigma, \mathcal{O}_\Sigma))$ is an analytic set of dimension at most one.*

It turns out that the Martinet surface contains all singular horizontal paths, that is, if $x \in M$ then $\mathcal{X}^x_\Delta \subset \Sigma$. This result follows from a characterization of singular paths in the cotangent bundle T^*M:

Proposition 6.2.4 (See Proposition 1.11 of [Ri14])**.** *Suppose that $M = \mathbb{R}^3$, that Δ is globally generated by Span$(X^1, \dots, X^k)$, and let (x, p) be the standard coordinate system of T^*M. The control $u \in \mathcal{U}^x$ is singular if and only if there exists an absolutely continuous path $\psi : [0,1] \to T^*M$ that never intersects the zero section of T^*M, such that*

$$h_i(\psi(t)) \equiv 0, \quad a.e. \ t \in [0,1]$$

where $h_i = X^i \cdot p$ and

$$\dot{\psi}(t) = \sum_{i=1}^{k} u_i(t) \vec{h}^i(\psi(t))$$

where $\vec{h}^i(\psi(t))$ is the Hamiltonian vector-field (in respect to the standard simplectic form) associated with the function $h_i(x,p)$.

Indeed, it now follows from a simple computation (which we omit, but can be found in [Ri14, Example 1.17 p. 27]) that, in the three dimensional case, the projection of $(h_i = 0)$ to M is the Martinet surface Σ. In particular, all singular horizontal paths must be contained in Σ. Since Σ has Hausdorff dimension at most 2, see Remark 6.2.2, this observation implies that the general Sard Conjecture holds true. Nevertheless, the following two stronger Conjectures are expected:

Sard Conjecture in three dimensions. *For every $x \in M$, the two-dimensional Hausdorff measure of $\mathcal{X}^x_\Delta$ is zero.*

Strong Sard Conjecture in three dimensions. *For every $x \in M$, the set $\mathcal{X}^x_\Delta$ has Hausdorff dimension at most* 1*.*

The strong version of the Conjecture is the best possible result one can hope for. The validity of these Conjectures are supported by the influential work of Zelenko and Zhitomirskii [ZZ95], where they prove the Strong version of the Conjecture for generic (with respect to the C^∞ Whitney topology) distributions Δ. In particular, in their paper the Martinet surface is always assumed to be smooth.

6.3 The Sard Conjecture over Martinet surfaces [BdSR18]

In collaboration with L. Rifford, we observed that the divergence of the vector field which generates the trace of the distribution Δ on Σ is controlled by its norm, see Proposition 6.7.2 below. To our knowledge, such an observation had not been made nor used before[2]. Combined with geometric-measure theory arguments, we obtain the following result which proves the Sard Conjecture whenever the Martinet surface is smooth:

Theorem 6.3.1 (Belotto da Silva and Rifford [BdSR18]). *Let M be a smooth manifold of dimension 3 and Δ a rank-two nonholonomic distribution on M whose Martinet surface Σ is smooth. Then for every $x \in M$ the set $\mathcal{X}^x$ has 2-dimensional Hausdorff measure zero.*

In the same paper, we consider the case when Σ is singular, under the extra assumption that it is real-analytic. We recall that under this assumption Σ has the structure of a coherent analytic space $(\Sigma, \mathcal{O}_\Sigma)$ and its singular set $\text{Sing}((\Sigma, \mathcal{O}_\Sigma))$ is an analytic curve. In the next Theorem, we follow the convention that $T_x\text{Sing}((\Sigma, \mathcal{O}_\Sigma)) = 0$ if x is a singular point of $\text{Sing}((\Sigma, \mathcal{O}_\Sigma))$. Now, combining our previous methods with resolution of singularities, we prove the following result:

Theorem 6.3.2 (Belotto da Silva and Rifford [BdSR18]). *Let M be an analytic manifold of dimension 3 and Δ be an analytic rank-two nonholonomic distribution on M. Assume that*

$$\Delta(x) \cap T_x Sing((\Sigma, \mathcal{O}_\Sigma)) = T_x Sing((\Sigma, \mathcal{O}_\Sigma)) \quad \forall x \in T_x Sing((\Sigma, \mathcal{O}_\Sigma)).$$

Then for every $x \in M$ the set $\mathcal{X}^x$ has 2-dimensional Hausdorff measure zero.

We will not discuss the ideas used in the proof of the above Theorem in this HDR, because we have made substantial progress in our later work

[2] Although resolution of singularities does not seem to have a direct correlation to this observation, I would like to point out that we noted this condition because it is a differential constraint which is preserved by blowings-up and resolution of singularities.

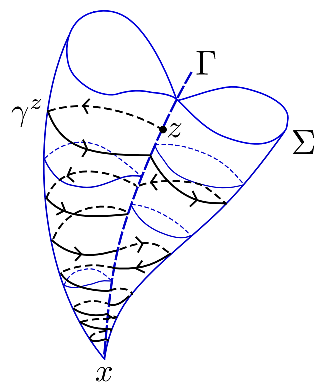

Figure 6.1: Figure 1 of [BdSFPR18].

[BdSFPR18], recovering a more enlightening proof of the above result. Indeed, our original proof was supported by precise computations involving the transform of vector-fields and metrics under blowings-up, while our new proof relies in a qualitative analysis of the transform of differential forms under resolution of singularities. Let us finish the discussion of the techniques used in [BdSR18] by illustrating the type of behavior which we **could not** control using them:

Exemple 6.3.2.1. *Consider the case when the Martinet surface Σ is stratified by a singleton $\{x\}$, a stratum Γ of dimension 1, and two strata of dimension 2 as in Figure 6.1. This occurs, for example, when:*

$$\Sigma = \Big\{h(x) := x_1^2 - x_2^2(x_3^2 - x_2^2) = 0\Big\}, \quad \Gamma = \Big\{x_1 = x_2 = 0, x_3 \neq 0\Big\}, \quad x = 0.$$

that is, Σ is a cone centered at $x = (0,0,0)$, with axis Γ and base equal to a spiric section curve $\{x_1^2 - x_2^2(1 - x_2^2) = 0\}$. The above equations are the Martinet surface of the nonholonomic distribution Δ generated by:

$$X^1 = \partial_{x_1}, \qquad X^2 = \partial_{x_2} + \big(x_1^3/3 - x_1 x_2^2(x_3^2 - x_2^2)\big)\partial_{x_3}.$$

Note that the distribution Δ does not satisfy the hypothesis of Theorem 6.3.2; indeed, $Sing((\Sigma, \mathcal{O}_\Sigma)) = \Gamma \cup \{x\}$ and, for every point $z \in \Gamma$:

$$\Delta(z) \cap T_z\Gamma = Span(\partial_{x_1}, \partial_{x_2}) \cap Span(\partial_{x_3}) = (0) \neq T_z\Gamma$$

Now, the characteristic vector-field is given by:

$$\mathcal{Z} = X^1(h)X^2 - X^2(h)X^1.$$

and we claim that its trajectories are (qualitative) as in figure 6.1. Indeed, we start by noting that $X^1(h) = 2x_1$ *and* $X^2(h) = 2x_2^2 + o(x_1, h)$*, implying that its singular set is equal to* $\Gamma \cup \{x\}$*. Next, note that:*

$$\mathcal{Z}(x_3) = -\frac{4}{3}x_1^4 + 2x_1^2 h(x)$$

implying that all trajectories contained in Σ *are non-increasing in respect to* $f(x) = x_3$*, proving the claim.*

In this kind of situation, from each z *in* Γ *there may exist* uncountable *many distinct trajectories between* x *and* z*. Indeed, there always exists two distinct leaves whose closure contains* z*, one in the right, and one on the left. Each one of these leaves intersect* Γ*, once again, at a point which is closer to* x*. Re-iterating this process, we obtain possibly* $2^{\mathbb{N}}$ *different concatenation of leaves connecting* z *to* x*. This is in sharp contrast with the study performed in [BdSR18], where the extra hypothesis of Theorem 6.3.2 guarantees that the line foliation is non-dicritical and, thus, we only need to consider an unique trajectory between* x *and* z*.*

6.4 Interlude: Semi-analytic sets

We briefly recall the definition of semianalytic sets used in the Theorem 6.5.1 below; for more details we refer the reader to [BM88].

Let M be a real analytic manifold of dimension n. A subset X of M is semianalytic if each $y \in M$ has a neighborhood U such that $X \cap U$ is a finite union of sets of the form

$$\Big\{x \in U \mid f_1(x) = \cdots = f_p(x) = 0, g_1(x) < 0, \ldots, g_l(x) < 0\Big\},$$

where $f_i : U \to \mathbb{R}$ and $g_j : U \to \mathbb{R}$ are real analytic functions. Note that we can always assume that $p = 1$ because we can always consider the sum of squares of the functions $f_i(x)$. In what follows, a semianalytic curve denotes any compact connected semianalytic subset of M of Hausdorff dimension at most 1.

6.5 The strong Sard Conjecture and regularity of geodesics for analytic sub-Riemannian structures in dimension 3 [BdSFPR18]

In collaboration with Figalli, Parusiński and Rifford, we have made substantial progress by improving our previous techniques and combining them with a Stokes type argument, which is essential for treating the phenomena illustrated in example 6.3.2.1. This allows us to provide a complete proof of the Strong Sard Conjecture for analytic manifolds of dimension three:

Theorem 6.5.1 ([BdSFPR18] Belotto da Silva, Figalli, Parusiński and Rifford)**.** *Let M be an analytic manifold of dimension 3 and Δ a rank-two nonholonomic analytic distribution on M. Then any singular horizontal curve is a semianalytic curve in M. Moreover, if g is a complete smooth Riemannian metric on M then, for every $x \in M$ and every $L > 0$, the set*

$$\mathcal{X}_{\Delta,g}^{x,L} := \{y \in \mathcal{X}_{\Delta}^{x};\, \exists \textit{ sing. hor. curve } \gamma, \gamma(0) = x,\, \gamma(1) = y, \textit{length}_g(\gamma) \leq L\}$$

is a finite union of singular horizontal curves, so it is a semianalytic curve. In particular, for every $x \in M$, the set $\mathcal{X}_{\Delta}^{x}$ is a countable union of semianalytic curves and it has Hausdorff dimension at most 1.

The above result allows us to address another main open problem in sub-Riemannian geometry, namely the regularity of length-minimizing horizontal paths. Indeed, combining our Theorem with the main result of [HL16], we obtain the first C^1 regularity result for singular minimizing geodesics in arbitrary analytic 3-dimensional sub-Riemannian structures. More precisely:

Corollary 6.5.2 ([BdSFPR18] Belotto da Silva, Figalli, Parusiński and Rifford)**.** *Let M be an analytic manifold of dimension 3, Δ a rank-two nonholonomic analytic distribution on M, and g a complete smooth sub-Rieman- nian metric over Δ. Let $\gamma : [0,1] \to M$ be a singular minimizing geodesic. Then γ is of class C^1 on $[0,1]$. Furthermore $\gamma([0,1])$ is semianalytic, and therefore it consists of finitely many points and finitely many analytic arcs.*

6.6 Characteristic line foliation

The local models given in Remark 6.2.1 have been explored, for example, in [ZZ95] and later in our work [BdSR18] in order to construct a locally-defined vector-field whose dynamics characterizes singular horizontal paths at almost every point. Since Σ admits a structure of coherent analytic space (see Remark 6.2.3 and c.f. §3.1), these local constructions yield a line foliation $\mathscr{L}$ (see §3.4), which we call *characteristic line foliation*, following Zelenko and Zhitomirskii [ZZ95, Section 1.4]. More precisely:

Lemma 6.6.1 (Characteristic line foliation)**.** *The set*

$$S := \Big\{ p \in \Sigma \,|\, p \in Sing(\Sigma) \text{ or } T_p\Sigma \subset \Delta_p \Big\}$$

(where $Sing(\Sigma)$ stands for the singularities of Σ as a space), is analytic of dimension less than or equal to 1, and there exists a line foliation $\mathscr{L}$ defined over Σ such that:

(i) The line foliation $\mathscr{L}$ is regular everywhere in $\Sigma \setminus S$.

(ii) If a horizontal path $\gamma : [0,1] \to M$ is singular with respect to Δ, then its image $\gamma([0,1])$ is contained in Σ and it is tangent to $\mathscr{L}$ over $\Sigma \setminus S$, that is

$$\gamma(t) \in \Sigma \setminus S \quad \Longrightarrow \quad \dot{\gamma}(t) \in \mathscr{L}_{\gamma(t)} \qquad \text{for a.e. } t \in [0,1].$$

Remark 6.6.2 (Characteristic vector-field)**.** *In the notation of Remark 6.2.1(ii), let h be a reduced analytic function whose zero set is equal to the Martinet surface Σ. Consider the vector-field*

$$\mathcal{Z} := X^1(h)X^2 - X^2(h)X^1.$$

Then the restriction of $\mathcal{Z}$ to Σ is a generator of the line foliation $\mathscr{L}$.

6.7 Key observation: Divergence of the characteristic vector-field

We start by considering a nonsingular analytic surface $\mathscr{S}$ (which plays the role of either a smooth Martinet surface Σ, or the resolution of singularities of Σ) with a volume form $\omega_{\mathscr{S}}$. Denote by $\mathcal{O}_{\mathscr{S}}$ the sheaf of analytic functions over $\mathscr{S}$. We note that there exists a one-to-one correspondence between differential 1-forms $\eta \in \Omega^1(\mathscr{S})$ and vector fields $\mathcal{Z} \in Der_{\mathscr{S}}$:

$$\mathcal{Z} \longleftrightarrow \eta \qquad \text{if } \eta = i_{\mathcal{Z}}\omega_{\mathscr{S}}.$$

This correspondence gives the following formula on the divergence:

$$\mathrm{div}_{\omega_{\mathscr{S}}}(\mathcal{Z})\omega_{\mathscr{S}} = d\eta.$$

We denote by $\mathcal{Z}(\mathcal{O}_{\mathscr{S}})$ the ideal sheaf generated by the derivation $\mathcal{Z}$ applied to the analytic functions in $\mathcal{O}_{\mathscr{S}}$, that is, the ideal sheaf locally generated by the coefficients of $\mathcal{Z}$. In what follows, we study closely the following property

Definition 6.7.1 (Key property)**.** *The vector-field $\mathcal{Z}$ has controlled divergence if*

$$div_{\omega_{\mathscr{S}}}(\mathcal{Z}) \in \mathcal{Z}(\mathcal{O}_{\mathscr{S}}).$$

Note that this property is independent of the volume form. Indeed, if $\omega_{\mathscr{S}}$ and $\omega'_{\mathscr{S}}$ are two volume forms over $\mathscr{S}$, then there exists a non-zero function such that $\omega'_{\mathscr{S}} = F \cdot \omega_{\mathscr{S}}$ and we verify easily that

$$\mathrm{div}_{\omega'_{\mathscr{S}}}(\mathcal{Z}) \cdot \omega'_{\mathscr{S}} = d\left(i_{\mathcal{Z}}\omega'_{\mathscr{S}}\right) = d\left(F \cdot i_{\mathcal{Z}}\omega_{\mathscr{S}}\right) = \left[\mathcal{Z}(F)/F + \mathrm{div}_{\omega_{\mathscr{S}}}(\mathcal{Z})\right]\omega'_{\mathscr{S}},$$

We now start the study over the Martinet surface Σ. Suppose that M is a 3-dimensional analytic manifold and denote by ω_M its volume form. Let $\delta \in \Omega^1(M)$ be an everywhere non-singular analytic 1-form which generates Δ, and denote by h the analytic function whose zero locus is the Martinet surface (c.f. equation (6.1)). We consider an analytic map $\pi : \mathscr{S} \to \Sigma \subset M$ from a smooth analytic surface $\mathscr{S}$ to the Martinet surface Σ, and we set $\eta := \pi^*(\delta)$. Let $\mathcal{Z}$ be the vector field associated to η, and denote by $\mathcal{Z}(\pi)$ the ideal subsheaf of $\mathcal{Z}(\mathcal{O}_{\mathscr{S}})$ generated by the derivation $\mathcal{Z}$ applied to the pullback by π of analytic functions on M. By using elementary arguments involving differential forms and the Hodge star operator, we get:

Proposition 6.7.2 (Divergence bound)**.** *The vector field $\mathcal{Z}$ satisfies the property $div_{\omega_{\mathscr{S}}}(\mathcal{Z}) \in \mathcal{Z}(\pi)$. In particular, for every compact subset $\mathcal{K} \subset \mathscr{S}$ there is a constant $K > 0$ such that*

$$|div_{\omega_{\mathscr{S}}}(\mathcal{Z})| \leq K\,|\pi_*(\mathcal{Z})| \qquad \text{on } \mathcal{K}.$$

In particular, a characteristic vector field always has controlled divergence. We finish this section recalling how the divergence control restricts the class of singular points of the line foliation:

Lemma 6.7.3 (Final Singularities)**.** *Let $\mathcal{Z}$ be a real analytic vector-field defined in an open neighborhood $U \subset \mathbb{R}^2$ of the origin and ω_U be a volume form over U. Let (x, y) be a coordinate system defined over U and suppose that:*

(i) The vector field $\mathcal{Z}$ has controlled divergence, that is,

$$div_{\omega_U}(\mathcal{Z}) \in \mathcal{Z}(\mathcal{O}_U);$$

(ii) There exists α and $\beta \in \mathbb{N}$ such that $\mathcal{Z} = x^\alpha y^\beta \, \widetilde{\mathcal{Z}}$, where $\widetilde{\mathcal{Z}}$ is either regular, or its singular points are isolated elementary singularities (see definition 3.5.1).

Then the vector field $\widetilde{\mathcal{Z}}$ is tangent to the set $\{x^\alpha y^\beta = 0\}$ and all of its singularities are saddles.

Sketch of proof. Let $m = x^\alpha y^\beta$. It follows from a direct computation that:

$$\left(\frac{\widetilde{\mathcal{Z}}(m)}{m} + \mathrm{div}_{\omega_U}(\widetilde{\mathcal{Z}}) \right) \in \widetilde{\mathcal{Z}}(\mathcal{O}_U)$$

It follows that the above expression has no poles, so $\widetilde{Z}$ must be tangent to $(m = 0)$. Next, denoting by λ and μ the eigenvalues of $\widetilde{Z}$ (which we assume, for simplicity, to correspond to the x and y directions), it is possible to obtain from the above expression evaluated at the origin that:

$$\lambda \cdot (\alpha + 1) + \mu(\beta + 1) = 0$$

Under the assumption that either λ or μ have non-zero real parts, we easily conclude that their real parts have opposite signs. The Lemma easily follows. □

6.8 Sketch of proof when the Martinet surface is non-singular

We are ready to sketch the proof of Theorem 6.5.2. We suppose by contradiction that there exists $x_0 \in M$ such that $\mathcal{X}^{x_0}_{\Delta} \subset \Sigma$ has positive two-dimensional Hausdorff measure. Apart from a geometrical measure theory argument (see [BdSR18, Lemma 3.1]) we can reduce the proof to a local statement. More precisely, we can suppose that:

− $M = U \subset \mathbb{R}^3$ is an open and connected set and g is the Euclidean metric;
− x belongs to the singular locus of the line foliation $\mathrm{Sing}(\mathscr{L})$;
− The line foliation $\mathscr{L}$ is generated by a globally defined (over U) vector field $\mathcal{Z}$;
− There exists a sub-set $S_0 \subset \Sigma \setminus \mathrm{Sing}(\mathscr{L})$ of positive two-dimensional Hausdorff measure;
− There exists a sub-set $S_\infty \subset \mathrm{Sing}(\mathscr{L})$ of Hausdorff dimension smaller or equal to 1;
− For every $z \in S_0$, there exists a half orbit ω_z of $\mathcal{Z}$ which is contained in U such that $\mathrm{length}(\omega_z) \leq 1$ and $\partial\omega_z \in S_\infty$.

Now, let $\varphi_t(z)$ denote the flow of $\mathcal{Z}$ at time t and initial condition z. Apart from changing the orientation of $\mathcal{Z}$ and making a standard measure theory argument, we can suppose that $\lim_{t\to\infty} \varphi_t(z) \in S_\infty$ for every $z \in S_0$. This implies that the sets $S_t := \varphi_t(S_0)$ are well-defined for every $t \in [0, \infty[$. Denote by vol^Σ the volume associated with the Euclidean metric on U. Since S_∞ has volume zero, by the dominated convergence Theorem, we get that:

$$\lim_{t\to\infty} \mathrm{vol}^\Sigma(S_t) = 0$$

Moreover, for every $z \in S_0$ and every $t \geq 0$, we have (where $|\cdot|$ denotes the Euclidean norm):

$$\int_0^t |\mathcal{Z}(\varphi_s(z))| ds \leq \mathrm{length}(\omega_z) \leq 1$$

Therefore, by Proposition 6.7.2 (and, formalizing the case that S_0 and S_t are not open sets via [BdSR18, Proposition B.1]), there exists $K > 0$ such that for every $t \geq 0$:

$$\begin{aligned}
\mathrm{vol}^\Sigma(S_t) &= \int_{S_t} \mathrm{vol}^\Sigma = \int_{S_0} \varphi_t^* \mathrm{vol}^\Sigma \\
&= \int_{S_0} \exp\left(\int_0^t \mathrm{div}^\Sigma_{\varphi_s(z)}(\mathcal{Z}) ds \right) \mathrm{vol}^\Sigma \\
&\geq \int_{S_0} \exp\left(-K \int_0^t |\mathcal{Z}(\varphi_s(z))| ds \right) \mathrm{vol}^\Sigma \\
&\geq e^{-K} \mathrm{vol}^\Sigma(S_0)
\end{aligned}$$

and, since this bound is independent of t, we get that

$$\lim_{t\to\infty} \mathrm{vol}^\Sigma(S_t) \geq e^{-K} \mathrm{vol}^\Sigma(S_0) > 0,$$

a contradiction.

6.9 Key result: resolution of singularities of the characteristic foliation

The treatment of the Sard Conjecture for singular Martinet surfaces is much more delicate than for the smooth case, essentially for the following two reasons. First, there is no divergence bound as in Proposition 6.7.2, unless the surface is smooth. This was already remarked in [BdSR18], where we studied the divergence property after a resolution of singularities of Σ. Second, the phenomena illustrated in Example 6.3.2.1 shows how transcendental the dynamics can be. In particular, it can not be treated via divergence arguments.

In order to deal with these issues, we provide a complete resolution of singularities of the Martinet surface Σ, the characteristic foliation $\mathscr{L}$, and the induced metric (up to bi-Lipschitz equivalence), in order to control the topological type of $\mathscr{L}$ and its relationship with the metric (in order to identify infinite-length horizontal paths). More precisely, the following is a key technical result of [BdSFPR18]:

Theorem 6.9.1. *Suppose that M and Δ are analytic, and consider a smooth Riemannian metric g over M. There exists a resolution of singularities*

$$\pi : (\mathscr{S}, E) \to (\Sigma, Sing(\Sigma))$$

of the Martinet surface Σ (where we recall that $\mathscr{S}$ is smooth, E is a SNC divisor and π is a locally finite sequence of compatible blowings-up) such that:

(I) (Reduction of the vector-field) Denote by $\widetilde{\mathscr{L}}$ the strict transform of the foliation $\mathscr{L}$ (whose definition is recalled in §3.5). Then all singularities of $\widetilde{\mathscr{L}}$ are saddle points.

(II) (Dicritical and non-dicritical components) The exceptional divisor E is given by the union of two locally compact sets of SNC divisors[3] E_{tan} and E_{tr}, where $E_{tan} \cap E_{tr}$ is a locally finite set of points, such that $\widetilde{\mathscr{L}}$ is tangent to E_{tan} and everywhere transverse to E_{tr}. Furthermore, the log-rank of π (that is, the rank on respect to logarithmic derivatives relative to E) over $E_{tr} \setminus E_{tan}$ is constant equal to 1.

(III) (Normal forms) At each point $\bar{z} \in E_{tan}$, there exists an open neighbourhood $U_{\bar{z}}$ of $\bar{z}$ such that:

[3] In the literature, it is common to call E_{tan} the non-dicritical divisor and E_{tr} the dicritical divisor.

(i) (1-points) Suppose that there exists only one irreducible component of E_{tan} passing through $\bar{z}$. Then there exists a coordinate system (u,v) centered at $\bar{z}$ and defined over $U_{\bar{z}}$, where $E_{tan} = (u = 0)$, such that:

(a) (Transition maps) Either $\bar{z}$ is a saddle point of $\widetilde{\mathscr{L}}$; or at each half-plane (bounded by E_{tan}) there exist two smooth analytic semi-segments $\Lambda^1_{\bar{z}}$ and $\Lambda^2_{\bar{z}}$ which are transverse to $\widetilde{\mathscr{L}}$ and E_{tan}, such that the flow (of a local generator $\widetilde{\mathscr{Z}}$) associated to $\widetilde{\mathscr{L}}$ gives rise to a bi-analytic transition map

$$\phi_{\bar{z}} : \Lambda^1_{\bar{z}} \to \Lambda^2_{\bar{z}},$$

and there exists a rectangle $V_{\bar{z}}$ bounded by E_{tan}, $\Lambda^1_{\bar{z}}$, $\Lambda^2_{\bar{z}}$ and a regular leaf $\mathcal{L} \not\subset E_{tan}$ of $\widetilde{\mathscr{L}}$ such that $\bar{z} \in \partial V_{\bar{z}} \setminus (\Lambda^1_{\bar{z}} \cup \Lambda^2_{\bar{z}} \cup \mathcal{L})$.

(b) (Transition maps compatible with E_{tr}) If $\bar{z} \in E_{tan} \cap E_{tr}$, furthermore, then $\bar{z}$ is a regular point of $\widetilde{\mathscr{L}}$ and $E_{tr} \cap U_{\bar{z}} = \{v = 0\}$ does not intersect $\Lambda^1_{\bar{z}}$ nor $\Lambda^2_{\bar{z}}$. Furthermore, the map $\phi_{\bar{z}}$ is the composition of two analytic maps:

$$\phi^1_{\bar{z}} : \Lambda^1_{\bar{z}} \to E_{tr}, \qquad \phi^2_{\bar{z}} : E_{tr} \to \Lambda^2_{\bar{z}}.$$

(ii) (2-points) Suppose that there exists two irreducible components of E_{tan} passing through $\bar{z}$. Then there exists a coordinate system $\boldsymbol{u} = (u_1, u_2)$ centered at $\bar{z}$ and defined over $U_{\bar{z}}$, where $E_{tan} = (u_1 \cdot u_2 = 0)$, such that:

(a) (Transition maps) At each quadrant (bounded by E_{tan}) there exists two smooth analytic semi-segments $\Lambda^1_{\bar{z}}$ and $\Lambda^2_{\bar{z}}$ which are transverse to $\widetilde{\mathscr{L}}$ and to E_{tan}, such that the flow (of a local generator $\widetilde{\mathscr{Z}}$) associated to $\widetilde{\mathscr{L}}$ gives rise to a bijective (but not necessarily analytic) transition map

$$\phi_{\bar{z}} : \Lambda^1_{\bar{z}} \to \Lambda^2_{\bar{z}}$$

and there exists a rectangle $V_{\bar{z}}$ bounded by E_{tan}, $\Lambda^1_{\bar{z}}$, $\Lambda^2_{\bar{z}}$ and a regular leaf $\mathcal{L} \not\subset E$ of $\widetilde{\mathscr{L}}$ such that $\bar{z} \in \partial V_{\bar{z}} \setminus (\Lambda^1_{\bar{z}} \cup \Lambda^2_{\bar{z}} \cup \mathcal{L})$.

(b) (Normal form compatible with the metric) There exists (α, β) in $\mathbb{N}^2$ such that the pulled-back metric $\pi^*(g)$ is locally bi-Lipschitz equivalent to:

$$h_{\bar{z}} = (d\boldsymbol{u}^\alpha)^2 + (d\boldsymbol{u}^\beta)^2, \text{ where } \boldsymbol{u}^\alpha = u_1^{\alpha_1} u_2^{\alpha_2} \text{ and } \boldsymbol{u}^\beta = u_1^{\beta_1} u_2^{\beta_2}.$$

Furthermore, there exists a vector field $\widetilde{\mathcal{Z}}$*, which locally generates* $\widetilde{\mathscr{L}}$*, such that:*
- either $|\widetilde{\mathcal{Z}}(\boldsymbol{u}^\alpha)| \geq |\widetilde{\mathcal{Z}}(\boldsymbol{u}^\beta)|$ *everywhere over* $V_{\bar{z}}$*, and* $\widetilde{\mathcal{Z}}(\boldsymbol{u}^\alpha) \neq 0$ *everywhere over* $V_{\bar{z}} \setminus E$*;*
- or $|\widetilde{\mathcal{Z}}(\boldsymbol{u}^\beta)| \geq |\widetilde{\mathcal{Z}}(\boldsymbol{u}^\alpha)|$ *everywhere over* $V_{\bar{z}}$*, and* $\widetilde{\mathcal{Z}}(\boldsymbol{u}^\beta) \neq 0$ *everywhere over* $V_{\bar{z}} \setminus E$*.*

The proof of this result involves a combination of resolution of singularities of surfaces (see §3.2), reduction of planar line-foliations (see §3.5), and Hsiang-Pati coordinates [BdSBGM17] (see §4.2). In order to prove point (II) above, furthermore, it is necessary to analyze the final singularities of $\widetilde{\mathscr{L}}$ under the differential constraint given by Proposition 6.7.2. This result allow us to describe in details the singular set $\mathcal{X}^x$ as we indicate in the next sections.

6.10 Complete description of the singular horizontal paths in dimension 3.

All of the results in this section, besides Proposition 6.10.6, follow from Theorem 6.9.1 and we omit their proofs. We note that all of these results, including Proposition 6.10.6, follow the philosophy used in the treatment of the Dulac Problem, from [Du1923] to [I91].

We start by obtaining a stratification of the Martinet surface Σ that is compatible with the distribution Δ (in [BdSFPR18], we prove this result via the more elementary method of Puisseux with parameters, in order to keep the presentation as simple as possible):

Lemma 6.10.1 (Stratification of Σ)**.** *There exists a regular semianalytic stratification of* Σ*,*

$$\Sigma = \Sigma^0 \cup \Sigma^1_{tr} \cup \Sigma^1_{tan} \cup \Sigma^2,$$

which satisfies the following properties:

(i) $S = \Sigma^0 \cup \Sigma^1_{tr} \cup \Sigma^1_{tan}$.

(ii) Σ^0 *is a locally finite union of points.*

(iii) Σ^1_{tan} *is a locally finite union of* 1*-dimensional strata with tangent spaces everywhere contained in* Δ*, that is,* $T_p\Sigma^1_{tan} \subset \Delta_p$ *for all* $p \in \Sigma^1_{tan}$.

(iv) Σ^1_{tr} *is a locally finite union of* 1*-dimensional strata transverse to* Δ *(that is,* $T_p\Sigma^1_{tr} \oplus \Delta_p = T_pM$ *for all* $p \in \Sigma^1_{tr}$*);*

(v) Σ^2 *is a locally finite union of* 2*-dimensional strata transverse to* Δ *(that is,* $T_p\Sigma^2 + \Delta_p = T_pM$ *for all* $p \in \Sigma^2$*).*

We note that, in the notation of Theorem 6.9.1, we have that $\pi(E_{tr} \setminus E_{tan}) \subset \Sigma^1_{tr}$ and $\pi(E_{tan}) \subset \Sigma^0 \cup \Sigma^1_{tan}$ (equality does not hold because real-analytic hypersurfaces do not have constant dimension, e.g. $(x^2 = yz^2)$). The proof of Theorem 6.5.1 now follows from analyzing, for every point $p \in \text{Sing}(\Sigma)$, the sets:

$$\mathcal{X}^{p,\epsilon}_{\Delta,g} = \{q \in \Sigma;\ \exists \text{ singular horizontal path from } p \text{ to } q \text{ of length } \leq \epsilon\},$$

where g is a smooth Riemannian metric over M. We start by noting that when $p \in \Sigma^1_{tr}$, then Theorem 6.9.1 implies that $\mathcal{X}^{p,\epsilon}_{\Delta,g}$ is a finite union of curves:

Lemma 6.10.2 (Local triviality of Δ along Σ^1_{tr})**.** *Let* Γ *be a* 1*-dimensional stratum in* Σ^1_{tr} *and let* $p \in \Gamma$ *be fixed. Then the following properties hold:*

(i) There exists a neighborhood $\mathcal{V}$ *of* p *and* $\delta > 0$ *such that, for every point* $q \in \mathcal{V} \cap \Sigma^1_{tr}$ *and every injective singular horizontal path* $\gamma : [0,1] \to \Sigma$ *such that* $\gamma(0) = q$*,* $\gamma(1) \in \Sigma^1_{tr}$*, and* $\gamma((0,1)) \subset \Sigma^2$*, the length of* γ *is larger than* δ*.*

(ii) The image of a singular horizontal path $\gamma : [0,1] \to M$ *such that* $\gamma([0,1)) \subset \Sigma^2$ *and* $\gamma(1) \in \Sigma^1_{tr}$ *is semianalytic.*

In particular, if $\mathcal{V}$ *is a neighbourhood of* p *in* M *such that* $\Sigma^2 \cap \mathcal{V}$ *is the disjoint union of the 2-dimensional analytic submanifolds* $\Pi_1, \ldots, \Pi_r$ *as in Lemma 6.10.1, then for* $\epsilon > 0$ *small enough there are singular horizontal paths* $\gamma_1, \ldots, \gamma_r : [0,1] \to \Sigma$ *with* $\gamma_i(0) = p$ *and* $\gamma_i((0,1]) \subset \Pi_i$ *for* $i = 1, \ldots, r$ *such that*

$$\mathcal{X}^{p,\epsilon}_{\Delta,g} = \bigcup_{i=1}^{r} \gamma_i\left([0,1]\right).$$

It remains to study the singular paths which intersect the set:

$$\widetilde{\Sigma} := \Sigma^0 \cup \Sigma^1_{tan}, \tag{6.2}$$

and it is enough to study (all) singular paths whose adherence contain a point of $\widetilde{\Sigma}$. Therefore, we can restrict our attention to a special type of trajectories of the characteristic foliation $\mathscr{L}$:

Definition 6.10.3 (Convergent transverse-singular trajectory)**.** *We say that an absolutely continuous path* $\gamma : [0,1) \to \Sigma$ *is a* transverse-singular trajectory *if*

$$\dot{\gamma}(t) \in \mathscr{L}_{\gamma(t)} \qquad \text{for a.e. } t \in [0,1),$$

and

$$\gamma(t) \in \Sigma^2 \cup \Sigma^1_{tr} \qquad \forall\, t \in [0,1).$$

Moreover, we say that γ *is* convergent *if it admits a limit in* $\widetilde{\Sigma}$ *as* t *tends to* 1.

Among these trajectories, it is interesting to consider the following dichotomy, inspired by the classification of trajectories of analytic vector fields:

Definition 6.10.4 (Characteristic and monodromic convergent transverse-singular trajectories)**.** *Let* $\gamma : [0,1) \to \Sigma$ *be a convergent transverse-singular trajectory such that* $\bar{y} := \lim_{t\to 1} \gamma(t)$ *belongs to* $\widetilde{\Sigma}$ *(see* (6.2)*). Then we say that:*

(i) γ *is* monodromic *if there exists a section* $\Lambda \subset \Sigma$ *of* $\mathscr{L}$ *at* $\bar{y}$ *such that* $\gamma([0,1)) \cap \Lambda$ *is the discrete and infinite union of points. In addition, we say that* γ *is* final *if* $\gamma([0,1)) \cap \Sigma^1_{tr}$ *is empty or infinite. In the latter case, we may choose as section* Λ *a branch of* Σ^1_{tr}.

(ii) γ *is* characteristic *if it is not monodromic.*

From now on, we call monodromic (resp. characteristic) trajectory any convergent transverse-singular trajectory with a limit in $\widetilde{\Sigma}$ which is monodromic (resp. characteristic). The following result is a direct consequence of Theorem 6.9.1, and the fact that the characteristic trajectories correspond, in the resolution space, to characteristics of an analytic vector field with singularities of saddle type.

Proposition 6.10.5. *Let* Σ^0 *and* $\widetilde{\Sigma}$ *be as in Lemma 6.10.1 and* (6.2)*. There exist a locally finite set of points* $\widetilde{\Sigma}^0$*, with* $\Sigma^0 \subset \widetilde{\Sigma}^0 \subset \widetilde{\Sigma}$*, such that the following properties hold:*

(i) If $\gamma : [0,1) \to \Sigma$ *is a convergent transverse-singular trajectory such that* $\bar{y} := \lim_{t\to 1} \gamma(t)$ *belongs to* $\widetilde{\Sigma}$ *then* $\bar{y}$ *belongs to* $\widetilde{\Sigma}^0$*. Moreover, if* γ *is characteristic then* $\gamma([0,1))$ *is semianalytic and there is* $\bar{t} \in [0,1)$ *such that* $\gamma([\bar{t},1)) \subset \Sigma^2$.

(ii) *For every $\bar{y} \in \widetilde{\Sigma}^0$ there exists only finitely many (possibly zero) characteristic trajectories converging to $\bar{y}$ and all of them are semianalytic curves.*

Note that if there are no monodromic trajectories, then the above Proposition guarantees that $\mathcal{X}^{p,\epsilon}_{\Delta,g}$ is everywhere a semi-analytic curve, proving Theorem 6.5.1. Unfortunately, there is no way to exclude mondromic trajectories (compare with Example 6.3.2.1), which are common. Fortunately, nevertheless, all of these trajectories have infinite length, and can not be realized by an horizontal path (which is an absolutely continuous curve with compact support):

Proposition 6.10.6 (Length of monodromic trajectories)**.** *The length of any monodromic trajectory is infinite.*

Note that these results provide a complete description of $\mathcal{X}^{p,\epsilon}_{\Delta,g}$, for every $p \in \mathrm{Sing}(\Sigma)$ and Theorem 6.5.1 easily follows.

The proof of Proposition 6.10.6 is the only result of this section that does not follow, in a simple way, from Theorem 6.9.1. Indeed, the proof has two steps and is done by contradiction. The first step consists in showing that if γ has finite length, then every monodromic trajectory which is "topologically equivalent" to γ also has finite length. This is done by combining Theorem 6.9.1 (in particular, part III.ii.b plays a key role) with a recent result about the regularity of transition maps [Sp18]. Hence, the assumption of finiteness on the length of γ implies that $\mathcal{X}^{\bar{y}}_{\Delta}$ has positive 2-dimensional Hausdorff measure. The second step consists in using an analytic argument in the cotagent sheaf, based on Stokes' Theorem, to obtain a contradiction. We refer the reader to [BdSFPR18, §4.4 and A] for details.

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