TD 1

Problems: Mathematical formalism

The objective of these exercises is for you to train how to write proofs (the exercises are not supposed to be difficult). Write the solutions as well as you can.

Problem 1. Prove that for every integer x, if x is odd then there exists an integer y such that $x^2 = 8y + 1$.

For Problems 2 and 3 you will need the following definition:

Definition 1. A function $f : \mathbb{R} \to \mathbb{R}$ is one-to-one (over all $\mathbb{R}$) if, $\forall x \in \mathbb{R}$ and $\forall y \in \mathbb{R}$:

$$\text{If } f(x) = f(y) \text{ then } x = y$$

Problem 2. Let $c \neq 0$ be a real number. Show that the function $f(x) = cx$ is one-to-one.

Problem 3. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be two one-to-one functions. Show that the composition $f \circ g : \mathbb{R} \to \mathbb{R}$ is one-to-one.

Problem 4. Fix a real number $x \neq 1$. Show (by induction) that for every non negative integer n,

$$1 + x + x^2 + \cdots + x^n = \frac{x^{n+1} - 1}{x - 1}.$$

Problems: Dedekind cuts

Let us extend the usual operations of addition, multiplication, and comparison from $\mathbb{Q}$ to $\mathbb{R}$ via Dedekind cuts. We recall that:

Definition 2 (Dedekind cut)**.** A subset $C \subset \mathbb{Q}$ is a Dedekind cut if:

- (Properness) The set C is neither $\emptyset$, nor equal to $\mathbb{Q}$.
- (Closed downwards) Let $p \in C$ and $q \in \mathbb{Q}$. If $q < p$ then $q \in C$.
- (No maximum) for all $p \in C$, there exists $q \in C$, such that $p < q$.

In the following, $+$, $\cdot$, $<$, and $\leq$ refer to addition, multiplication, and comparisons on *rational numbers*. In Problems 5 to 8, you may only use properties coming from the definition of Dedekind cuts, and properties of the above operations on rational numbers.

Problem 5. If x and y are Dedekind cuts, we defined in class their sum by

$$x \oplus y = \{p + q \mid p \in x \text{ and } q \in y\}.$$

Prove that $x \oplus y$ is also a Dedekind cut.

Note that $\oplus$ is commutative, i.e., $x \oplus y = y \oplus x$ for any x and y (you do not have to prove this, and you may assume it from now on).

Problem 6. Define $O = \{p \in \mathbb{Q} \mid p < 0\}$ (here 0 is zero in the rational numbers). In the lecture we have already shown that this is a Dedekind cut. Show that for any Dedekind cut x, we have $x \oplus O = x = O \oplus x$.

Recall that in lecture we defined a total order

$$x \preceq y \iff x \subseteq y$$

on Dedekind cuts (we are using the notation $(\preceq)$ right now to distinguish from $\leq$, which is the ordering only on $\mathbb{Q}$). We say that a Dedekind cut x is *nonnegative* if $O \preceq x$.

Problem 7. Define the product of two nonnegative Dedekind cuts x and y by the following:

$$x \odot y = O \cup \{p \cdot q \mid p \in x \setminus O \text{ and } q \in y \setminus O\}.$$

Show that $x \odot y$ is a Dedekind cut.

Problem 8. For a Dedekind cut x, define its negative as

$$\ominus x = \{-p \mid p \in \mathbb{Q} \setminus x \text{ and } p \text{ is not a minimum element of } \mathbb{Q} \setminus x\}.$$

(a) (Warmup) Show that $\ominus O = O$.
(b) (Useful technical step) Show that $\mathbb{Q} \setminus x$ is *closed upward*, i.e., if $p \in \mathbb{Q} \setminus x$ and $p' \in \mathbb{Q}$ with $p' > p$, then $p' \in \mathbb{Q} \setminus x$.
(c) Show that $\ominus x$ is a Dedekind cut.
(d) (Another technical step) For each $r < 0$ show that there exists $p' \in x$ with $p' - r \in \mathbb{Q} \setminus x$.
(e) Show that $x \oplus (\ominus x) = O$ for any Dedekind cut x.

Note that we can use this definition in order to extend multiplication to all pairs of Dedekind cuts; you do not have to do so here.