Notes about Dedekind cuts

André Belotto

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1 Introduction

In these notes, we assume that the field of rational numbers $\mathbb{Q}$ has already been constructed, that the (usual) operations of addition $+ : \mathbb{Q}^2 \to \mathbb{Q}$ and multiplication $\times : \mathbb{Q}^2 \to \mathbb{Q}$ are defined, as well as the usual 'less-than-or-equal' relation $\leq$. Note that this assumption does not belong to a set theory course: in such a course, we would start by defining the Zermelo-Fraenkel (set-theoretical) axioms, which allow the (set-theoretical) construction of the natural numbers $\mathbb{N}$. The construction of $\mathbb{Z}$ and $\mathbb{Q}$ would then follow from algebraic arguments.

Our objective is to *construct* the real numbers $\mathbb{R}$ and to properly define their sum and product operations. In other words, we want to answer questions such as:

- What does π mean? (in terms of the rational numbers $\mathbb{Q}$)

As we will see, answering this kind of question will demonstrate one of the most important differences between $\mathbb{R}$ and $\mathbb{Q}$: the "lack of topological holes" (that is, its *completeness*). This property is (at least, implicitly) used in most fundamental results in a real analysis course (e.g. the Intermediate Value Theorem, the Bolzano-Weiersstrass Theorem, the connectedness of $\mathbb{R}$, etc.). There are (at least) three classical ways to construct $\mathbb{R}$ from $\mathbb{Q}$:

1. Dedekind cuts;
2. Infinite decimal expansions;
3. Cauchy sequences.

Here, we have chosen the first approach. We note that (3) is usually chosen in an advanced course, but this is not suitable here: we want to motivate the notion of Cauchy sequences by construction of $\mathbb{R}$, and not the other way around. There is no discernable difference between (1) and (2).

2 Dedekind cuts

Intuitively, $\mathbb{Q}$ is the line with "holes" and $\mathbb{R}$ should be the line with *no* "holes". How are we going to fill in the blanks in $\mathbb{Q}$?

The idea of a Dedekind cut is, loosely, the following: consider a real number x which we would like to construct (that is, consider choosing a point in the 'number line'). We will represent this number via a set of rational numbers:

$$\text{“}x = (-\infty, x) \cap \mathbb{Q}\text{”}.$$

But this is, *of course*, not a good definition! We can not define x by using what x means. The rigorous (and correct!) way is the following.

Definition 2.1 (Dedekind cut)**.** A subset $C \subset \mathbb{Q}$ is a Dedekind cut if:

- (Properness) the set C is neither $\emptyset$ nor $\mathbb{Q}$;
- (Downwards closed) for all $p \in C$ and $q \in \mathbb{Q}$, if $q < p$ then $q \in C$;
- (No maximal element) for all $p \in C$, there exists $q \in C$, such that $p < q$.

Definition 2.2 (Real numbers)**.** The set of real numbers $\mathbb{R}$ is the set of all Dedekind cuts.

From now on, we will denote a Dedekind cut by x or y (instead of C). We still need to convince ourselves that the above definition of $\mathbb{R}$ does indeed recover the set that we are used to working with. In particular, let us show that there is a copy of $\mathbb{Q}$ contained in $\mathbb{R}$:

Example 2.3. Let $p \in \mathbb{Q}$ be fixed. The associated real number is

$$x_p = \{r \in \mathbb{Q};\ r < p\}.$$

First, let us prove that this set is a Dedekind cut.

- (Properness) Note that $p-1 \in \mathbb{Q}$ and $p-1 < p$, which implies that $x_p \neq \emptyset$. Furthermore, $p+1 \geq p$, which implies that $p+1 \notin x_p$ and thus $x_p \neq \mathbb{Q}$.
- (Downwards closed) Let $r \in x_p$ and pick some fixed $q \in \mathbb{Q}$ with $q < r$. We must show that $q \in x_p$. But $q < r < p$ implies that $q < p$, whence $q \in x_p$ by our definition of x_p.
- (No maximal element) We may argue either directly or by contradiction. Let us argue directly. Fix an arbitrary $r \in x_p$. Then, since $r < p$, we have that $0 < p - r$. Now, $p - r \in \mathbb{Q}$ and $0 < \frac{p-r}{2} \in \mathbb{Q}$. We claim that $q = r + \frac{p-r}{2}$ is bigger than r *and* that it belongs to x_p. Indeed,

$$r < r + \frac{p-r}{2} = \frac{p+r}{2} < \frac{p+p}{2} = p$$

Exercise 2.4. Prove that the set

$$x_{\sqrt{2}} = \{r \in \mathbb{Q}; r < 0 \text{ or } r^2 < 2\}.$$

is a Dedekind cut (which we claim represents $\sqrt{2}$).

Remark 2.5. We are not ready (yet) to define the the Dedekind cut which represents π, because π is a transcendental number.

We now need to define the main operations in $\mathbb{R}$ in order to show that it is an ordered field. In particular, we will define binary operations $(\oplus, \otimes)$ and an order $(\preceq)$.

Definition 2.6 (Order)**.** Let x and y be two Dedekind cuts. We define a order $\preceq$ by saying that $x \preceq y$ if and only if $x \subseteq y$.

Definition 2.7 (Sum)**.** If x and y are Dedekind cuts then we define their sum as

$$x \oplus y = \{p + q \mid p \in x \text{ and } q \in y\}.$$

(You will prove that this set is a Dedekind cut in the TDs).

Remark 2.8 (Product)*.* The product of Dedekind cuts will be defined and studied in the TDs.

It is, of course, necessary to show that these operations satisfy all of the usual properties we are used to (associativity, commutativity, etc.).

Proposition 2.9 (Basic Properties)**.** *The data* $(\mathbb{R}, \oplus, \otimes)$ *defines a field. Furthermore,* $(\preceq)$ *is a total order on* $\mathbb{R}$*, i.e., for all Dedekind cuts* x*,* y*, and* z*,*

- *(Reflexive and Antisymmetry) both* $x \preceq y$ *and* $y \preceq x$ *hold if and only if* $x = y$*;*
- *(Transitivity) if* $x \preceq y$ *and* $y \preceq z$ *then* $x \preceq z$*;*
- *(Comparability) either* $x \preceq y$ *or* $y \preceq x$*.*

Proof. You will prove that $(\mathbb{R}, \oplus, \otimes)$ is a field in the TD. The fact that $(\preceq)$ is reflexive, anti-symmetric, and transitive follows from the usual properties of $(\subseteq)$ (can you show this?). Therefore, let us turn to the proof of comparability.

Suppose that x is not contained in y. Then, by definition, there exists an element $p \in x$ such that $p \notin y$. We claim that, for every element $q \in y$, we have that $q < p$. Indeed, if there exists $q \in y$ such that $q \geq p$, then $p \in y$, since y is closed downwards, which gives a contradiction. Therefore, every element $q \in y$ is such that $q < p$. Once again, since x is closed downwards, we have that $q \in x$ for every $q \in y$. We conclude that $y \subseteq x$ (as sets), whence, by definition of $\preceq$, $y \preceq x$. ☐

Finally, note that $\mathbb{R}$ contains a copy of $\mathbb{Q}$ which "respects" the usual operations. That is, there is a homomorphism of fields $f : \mathbb{Q} \to \mathbb{R}$ given by $f : p \mapsto x_p = \{q \in \mathbb{Q} \mid q < p\}$ (that this function is indeed an homomorphism is an exercise) that respects the total orders, as formalised by the following lemma.

Lemma 2.10. *Let* $p, q \in \mathbb{Q}$*. If* $p < q$*, then* $f(p) \preceq f(q)$*.*

Proof. Let $p, q \in \mathbb{Q}$ be such that $p < q$. Let us show that $x_p \subseteq x_q$. Indeed, fix an arbitrary $r \in x_p$, which means that $r < p$. Since $r < p < q$ we have that $r < q$, and, by the 'downwards closed' property for x_q, we conclude that $r \in x_q$. Since the choice of $r \in x_p$ was arbitrary, $x_p \subseteq x_q$. ☐

3 Property of the real numbers: Supremum

You can find a good introduction of this topic in the book on which these notes are based. We now denote by $(\mathbb{R}, +, \times, \leq)$ the usual operations over $\mathbb{R}$ (instead of $(\mathbb{R}, \oplus, \otimes, \preceq)$). Let us go directly to the important definitions.

Definition 3.1. Let $S \subset \mathbb{R}$ be a proper subset.

- A number $M \in \mathbb{R}$ is said to be an *upper bound* for S if, for all $x \in S$, $x \leq M$.
- The set S is said to be *bounded above* if there exists an upper bound M of S.
- The supremum of S (*if it exists*) is the *smallest* upper bound of S, and is denoted by $\sup(S)$. Explicitly,[1]
  1. $\sup(S)$ is an upper bound of S;
  2. for all upper bounds M of S, $\sup(S) \leq M$.

  Note that there is no reason for the supremum $\sup(S)$ to be an element of S. If it *is*, then it is called the *maximum* $\max(S)$. (See the following examples.)

Note 3.2. Dually, there exist the notions of *infimum* and *minimum*, where the infimum is defined as the maximum of the set of lower bounds.

Example 3.3.

1. The set $[0, 1) = \{x \in \mathbb{R} \mid 0 \leq x < 1\}$ has minimum equal to 0 and supremum equal to 1 (but *no* maximum, since the supremum is not contained in the set itself).
2. The set $\{\frac{1}{n} \mid n \in \mathbb{N} \setminus \{0\}\}$ has maximum equal to 1 and a infimum (and no minimum) equal to 0. (As a small warning, note that some authors write $\mathbb{N}$ to mean $\{1, 2, \dots\}$, while others use it to denote $\{0, 1, 2, \dots\}$.)

Now, the basic theorem that guarantee there are no "holes" in the real numbers is the following.

Theorem 3.4. *Let $S \subset \mathbb{R}$ be a non-empty subset. If S is bounded above, then there exists a supremum of S.*

Proof. Each point $x \in S$ defines a Dedekind cut[2] and, therefore, a subset of $\mathbb{Q}$. We can take the union of all of these sets:

$$y = \bigcup_{x \in S} x = \{p \in \mathbb{Q} \mid \text{there exists } x_0 \in S \text{ such that } p \in x_0\}$$

so that, in particular, $y \subseteq \mathbb{Q}$. We claim that y is the supremum of S. In order for this to be true, we must verify that:

[1]Another way of saying this is that the supremum is the minimum element in the set of all upper bounds.

[2]Remember, an element $x \in \mathbb{R}$ is itself a set, and so statements like $x \in x' \in \mathbb{R}$ and $\bigcup_{x \in S} x$ make sense!

(i) y is a Dedekind cut;

(ii) y is an upper bound of S;

(iii) y is the smallest upper bound of S.

We start by proving (i):

- (Properness). Since $S \neq \emptyset$, there exists $x \in S$ and, therefore, $x \in y \neq \emptyset$.

  Next, since S is bounded above, there exists an upper bound M of S, i.e. for all $x \in S$, $x \subseteq M$. Let $p \notin M$ be any point. Then $p \notin x$ for any $x \in S$ (otherwise this would contradict the downwards closed property) and therefore $p \notin y$.

- (Downwards closed) Let $p \in y$, and let $q \in \mathbb{Q}$ be such that $q < p$. Since y is a union, $p \in x$ for some $x \in S$. By the 'downwards closed' property of x, $q \in x$ and, therefore, $q \in y$.

- (No maximal element) Let $p \in y$. Since y is a union, $p \in x$ for some $x \in S$. Since x is a Dedekind cut, there exists $q \in x$ such that $p < q$. It is clear that $q \in y$ and so we are done.

The proof of (ii) is tautological. Indeed, for every $x \in S$, it is clear that

$$x \subseteq \bigcup_{x \in S} x = y.$$

Finally, let us prove (iii). Let M be an upper bound of S, so that $x \subseteq M$ for all $x \in S$. This implies that

$$y = \bigcup_{x \in S} x \subseteq M$$

which finishes the proof. □

Example 3.5. We are ready to define the Dedekind cut which represents the number π. Indeed, the number $\pi = 3.141...$ is the supremum of the set of real numbers

$$S = \{3, 3.1, 3.14, 3.141, etc\}$$

which is bounded above by 4 (which shows that the supremum must exist). This set is exactly the increasing decimal expansions of π: first to zero decimal places, then to one, then two, and so on.

Alternatively, we could use a series representation of π, for example the Leibniz series. Then π is the supremum over $n \in \mathbb{N}$ of the sets

$$S_n = \left\{ p \in \mathbb{Q} \mid p < 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \cdots + \frac{4}{4n+1} - \frac{4}{4n+3} \right\}$$

i.e. $\pi = \sup_{n \in \mathbb{N}} S_n := \sup\{S_n \mid n \in \mathbb{N}\}$. Note that the last term is a subtraction, so that we end up with a number less than π.

Theorem 3.6 (Archimidean Principle)**.** *Let x and y be positive real numbers (we will properly define what it means to be positive in the TD). Then there exists $n \in \mathbb{N}$ such that $nx > y$.*

Proof. We prove this result by contradiction. Assume that there does not exist such an n, i.e. that for all $n \in \mathbb{N}$, $nx \leqslant y$. Now consider the set

$$S = \{nx \mid n \in \mathbb{N}\}.$$

This is bounded above by y, and so $\sup(S)$ exists, and $\sup(S) \leq y$ (since y is an upper bound). But, since x is positive:

$$sup(S) - x < sup(S)$$

and $\sup(S) - x$ is *not* an upper bound of S. So, there exists $n \in \mathbb{N}$ such that $\sup(S) - x \leq nx$, which implies that $\sup(S) < (n+1)x$. But this is a contradiction, since $\sup(S)$ is an upper bound of S. $\square$