# INNER LIPSCHITZ GEOMETRY OF COMPLEX SURFACES: A VALUATIVE APPROACH

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Abstract. Given a complex analytic germ $(X,0) \subset (C^n,0)$, the standard Hermitian metric of $\mathbb{C}^n$ induces a natural arc-length metric on $(X,0)$, called the inner metric. We study the inner metric structure of the germ of an isolated complex surface singularity $(X,0)$ by means of a family of natural numerical invariants, called inner rates. Our main result is a formula for the Laplacian of the inner rate function on a space of valuations, the non-archimedean link of $(X,0)$. We deduce in particular that the global data consisting of the topology of $(X,0)$, together with the configuration of a generic hyperplane section and of the polar curve of a generic plane projection of $(X,0)$, completely determines all the inner rates on $(X,0)$, and hence the local metric structure of the germ. Several other applications of our results are discussed in the paper.

## 1. Introduction

Given a complex analytic germ $(X,0) \subset (\mathbb{C}^n,0)$ at the origin of $\mathbb{C}^n$, the standard Hermitian metric of $\mathbb{C}^n$ induces a natural metric on $(X,0)$ by measuring the lengths of arcs on $(X,0)$. This metric, called the *inner metric*, has been widely studied in several different contexts. For example, in [BL07] it is used to introduce a notion of metric tangent cone, while in [HP85] and [Nag89] it is used to study the the $L^2$-cohomology of singular algebraic surfaces, following original ideas of [Che80]. In the case of surfaces, the isomorphism class of the metric germ $(X,0)$ modulo bi-Lipschitz homeomorphisms was the main object of study of [BNP14], where a complete classification is given.

The study of the inner metric is motivated by the following fact: while it is well known that, for $\epsilon > 0$ sufficiently small, $X$ is locally homeomorphic to the cone over its link $X^{(\epsilon)} = X \cap S_\epsilon$, where $S_\epsilon$ denotes the sphere centered at 0 with radius $\epsilon$ in $\mathbb{C}^n$, in general the metric germ $(X,0)$ is not metrically conical. Indeed, there are parts of its link $X^{(\epsilon)}$ whose diameters with respect to the inner metric shrink faster than linearly when approaching the singular point. It is then natural to study how $X^{(\epsilon)}$ behaves metrically when approaching the origin.

In this paper, we study the metric structure of the germ of an isolated complex surface singularity $(X,0)$ by means of a family of natural numerical invariants, called *inner rates*. Given a good resolution $\pi\colon X_\pi \to X$ of $(X,0)$ that factors through the blowup of the maximal ideal and through the Nash modification of $(X,0)$, an irreducible component $E$ of the exceptional divisor $\pi^{-1}(0)$ of $\pi$, and a small neighborhood $\mathcal{N}(E)$ of $E$ in $X_\pi$ with a neighborhood of each double point of $\pi^{-1}(0)$ removed, the inner rate $q_E$ of $E$ is a rational number that measures how fast the subset $\pi\big(\mathcal{N}(E)\big)$ of $(X,0)$ shrinks when approaching the origin (see Definition 3.3 for a precise definition). These invariants, which were originally introduced in [BNP14], where they were used to define a metric decomposition of the germ $(X,0)$, have the advantage that they are independent of the choice of an embedding of $(X,0)$ into a smooth germ, only depending on the analytic type of $(X,0)$. As the sets of the form $\pi\big(\mathcal{N}(E)\big)$ cover the germ $(X,0)$ and can be taken to be arbitrarily small by refining the resolution $\pi$, the knowledge of all the inner rates of the exceptional divisors of all good resolutions of $(X,0)$ gives a very fine understanding of the metric structure of the germ. For example, one can use the inner rates to compute the contact order for the inner metric of any pair of complex curve germs in $(X,0)$, using a minimax procedure (see Remark 3.4).

While the inner rates have raised significant interest in the last decade, it was still an open question to determine how the global geometry of the singularity $(X,0)$ influences their behavior. Our first main result states that the global data of the topology of $(X,0)$, together with the configuration of a generic hyperplane section and of the polar curve of a generic plane projection $:(X,0)\to(\mathbb{C}^2,0)$, not only influences the behavior of the inner rates, but in fact completely determines all of them.

**Theorem A.** *Let $(X,0)\subset(\mathbb{C}^n,0)$ be an isolated complex surface singularity, let $E$ be an exceptional prime divisor in some good resolution of $(X,0)$, and let $\pi\colon X_\pi\to X$ be the minimal good resolution of $(X,0)$ that factors through the blowup of the maximal ideal and through the Nash transform of $(X,0)$. Then the inner rate $q_E$ of $E$ is determined by (and can be computed in terms of) the following data:*

- *(i) the topological data consisting of the dual graph $\Gamma_\pi$ of $\pi^{-1}(0)$ decorated with the Euler classes and the genera of its vertices;*
- *(ii) on each vertex $v$ of $\Gamma_\pi$, an arrow for each irreducible component of a generic hyperplane section of $(X,0)$ whose strict transform on $X_\pi$ passes through the prime divisor $E_v$;*
- *(iii) on each vertex $v$ of $\Gamma_\pi$, an arrow for each irreducible component of the polar curve of a generic projection $:(X,0)\to(\mathbb{C}^2,0)$ whose strict transform on $X_\pi$ passes through the prime divisor $E_v$.*

In earlier papers on the subject, the inner rates were always computed by considering a generic projection $\ell\colon(X,0)\to(\mathbb{C}^2,0)$ and lifting the inner rates of the components of the exceptional divisor of a suitable resolution of the discriminant curve of $\ell$. Outside of the simplest examples, this approach turned out to be very impractical and computationally expensive, since it is generally very hard to compute discriminant curves and to decide whether a projection is generic. On the other hand, given the data of the theorem, which is generally much simpler to obtain, our result also provides a very easy way to compute the inner rate $q_E$ by means of an elementary linear algebra computation. To showcase this fact, in Example 5.5 we illustrate, for a fairly complicated singularity, how simple it is to obtain the inner rates, which were computed in [BNP14, Example 15.2] using Maple.

We obtain Theorem A as a consequence (see Corollary 5.1) of a stronger result about the Laplacian on the inner rate function on a space of valuations, a non-archimedean avatar of the link $X^{(\epsilon)}$ of $(X,0)$. Indeed, it is very natural to study the inner rates from the point of view of valuation theory, since $q_E$ only depends on the *divisorial valuation* associated with an exceptional divisor $E$.

The *non-archimedean link* $\mathrm{NL}(X,0)$ of $(X,0)$ is defined as the set of (suitably normalized) semi-valuations on the completed local ring $\mathcal{O}_{X,0}$ of $X$ at 0 that are trivial on $\mathbb{C}$ (see Definition 2.2). It is a Hausdorff topological space that contains the set of divisorial valuations of $(X,0)$ as a dense subset. For example, if $X$ is smooth at 0 then the associated non-archimedean link is a well known object, the valuative tree of [FJ04].

If $\pi\colon X_\pi \to X$ is a good resolution of $(X,0)$, then the dual graph $\Gamma_\pi$ of the exceptional divisor $\pi^{-1}(0)$ embeds naturally in $\mathrm{NL}(X,0)$, and the latter deforms continuously onto the former. These retractions allow us to see $\mathrm{NL}(X,0)$ as a universal dual graph, making it a very convenient object to study the inner rates.

We endow the dual resolution graphs $\Gamma_\pi$ with a metric defined as follows. Let $e=[v,v']$ be an edge of $\Gamma_\pi$ and let $E_v$ be the component of the exceptional divisor $\pi^{-1}(0)$ corresponding to $v$. Then the *multiplicity* $m_v$ of $v$ is defined as the order of vanishing on $E$ of the pullback on $X_\pi$ of a generic linear form $h$ of $(X,0)$ (that is, equivalently, $m_v$ is the multiplicity of $E_v$ in the exceptional divisor $\pi^{-1}(0)$). Similarly, denote by $m_{v'}$ the multiplicity of $v'$. Then we declare the length of the edge $e$ to be $1/(m_v m_{v'})$. This length has a natural geometrical interpretation as the opposite of the screw number of the representative of monodromy of $h$ defined as a product of Dehn twists on the annuli defined by the edge $e$. These lengths gives rise to a metric on $\Gamma_\pi$ and, by refining the resolution $\pi$, to a natural metric on $\mathrm{NL}(X,0)$.

The map sending a divisorial valuation $v$ to the associated inner rate $q_{E_v}$ extends canonically to a continuous function $\mathcal{I}_X\colon \mathrm{NL}(X,0)\to \mathbb{R}_+\cup\{+\infty\}$ which has the remarkable property that its restriction $\mathcal{I}_X|_{\Gamma_\pi}$ to any dual resolution graph $\Gamma_\pi$ is piecewise linear with integral slopes with respect to its metric.

This allows to study the inner rate function $\mathcal{I}_X$ using classical tools of potential theory on metric graphs, as is for example done in an arithmetic setting in [BN16]. Namely, our main result is a formula computing the *Laplacian* $\Delta_{\Gamma_\pi}\big(\mathcal{I}_X|_{\Gamma_\pi}\big)$ of the inner rate function on a dual resolution graph $\Gamma_\pi$, that is the divisor on $\Gamma_\pi$ whose coefficient $\Delta_{\Gamma_\pi}\big(\mathcal{I}_X|_{\Gamma_\pi}\big)(v)$ at a vertex $v$ of $\Gamma_\pi$ is the sum of the slopes of $\mathcal{I}$ on the (outgoing) edges of $\Gamma_\pi$ emanating from $v$ (see Section 2.3 for a detailed explanation of all the relevant notions).

In order to measure quantitatively the geometric data appearing in the statement of Theorem A, for any vertex $v$ of $\Gamma_\pi$ denote by $l_v$ (respectively by $p_v$) the number of arrows on $v$ associated with a generic hyperplane section (respectively with a generic polar curve) of $(X,0)$, and by $\check{E}_v$ the (reduced) curve obtained removing from $E_v$ the double points of $\pi^{-1}(0)$. We can now state our main result.

**Theorem B** (Laplacian of the inner rate function, see Theorem 4.2)**.** *let $\pi\colon X_\pi\to X$ be a good resolution of $(X,0)$ that factors through the blowup of the maximal ideal and through the Nash transform of $(X,0)$. For every vertex $v$ of $\Gamma_\pi$, we have:*

$$\Delta_{\Gamma_\pi}\big(\mathcal{I}_X|_{\Gamma_\pi}\big)(v) = m_v\big(2l_v - p_v - \chi(\check{E}_v)\big).$$

Besides the complete description of inner rates obtained in Theorem A, Theorem B has several interesting consequences. For example, as an immediate by-product of our result we deduce the fact that the inner rate function $\mathcal{I}^X$ is actually linear, and not just piecewise linear, along every string of the metric graph $\Gamma_\pi$ (Corollary 4.5). Furthermore, the fact that the Laplacian is a divisor of degree zero of $\Gamma_\pi$ (that is, the sum over all the vertices $v$ of $\Gamma_\pi$ of the integers $\Delta_{\Gamma_\pi}\big(\mathcal{I}_X|_{\Gamma_\pi}\big)(v)$ vanishes) allows us to retrieve the formula of Lê–Greuel type proven in [DGJ14] in the case of a generic linear form $L\colon (X,0)\to(\mathbb{C}^2,0)$.

The Laplacian formula of Theorem B also imposes several restriction on the possible configurations of the values of $l_v$ and $p_v$ on the vertices of the graph $\Gamma_\pi$. For example, fixing the inner rates on $\Gamma_\pi$ and the integers $l_v$ determines the localization of the strict transform of the polar curve $\Pi$ of a generic projection of $(X,0)$, and vice versa. Further restrictions on $l_v$ and $p_v$ are imposed by the fact that, even when we do not assume to know the inner rates of $(X,0)$, not every degree zero divisor can be realized as the Laplacian of a piecewise linear function; this is part of an rapidly developing research area and we refer the interested reader to the recent book [CP18] for an overview of divisor theory on metric graphs that focuses on similar questions. These facts can be interpreted as a first step towards the famous question of D. T. Lê (see [Lê00, Section 4.3] or [BL02, Section 8]) inquiring about the existence of a duality between the two main algorithms of resolution of a complex surface, via normalized blow-ups of points ([Zar39]) or via normalized Nash transforms ([Spi90]).

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Let us conclude this introduction by discussing several future directions whose exploration is made possible by our valuative point of view on inner rates, and which will be the subjects of further study by the authors.

First of all, as observed in Remark 3.10, the inner rates can be used to define a ultrametric (that is, non-archimedean) distance on $\mathrm{NL}(X,0)$ via a standard minimax procedure which in this case has a very natural geometric interpretation. Ultrametric distances on valuations spaces of surface singularities have been recently studied in [BPPPR18]; however only a special class of surface singularities, those having contractible non-archimedean link (called *arborescent* in *loc.cit.*) were endowed there with an ultrametric distance carrying a geometric interpretation. This non-archimedean inner distance on $\mathrm{NL}(X,0)$ will be used to define canonical metric decompositions of non-archimedean links, parallel to those of [BNP14] but much more intrinsic.

The inner rate of a divisorial valuation $v$ of $(X,0)$ can be studied via birational techniques, namely using (logarithmic) Fitting ideals. More precisely, it can be shown that the inner rate $\mathcal{I}_X(v)$ of $v$ equals the Mather discrepancy of $v$ (normalized by the multiplicity $m_v$) minus one. In particular, in the smooth case the inner rate function $\mathcal{I}_{\mathbb{C}^2}$ equals the normalized log discrepancy function on $\mathrm{NL}(\mathbb{C}^2,0)$, as studied for example in [BPPP18], minus one. This approach would be well suited to be generalized to the higher dimensional case, where the inner metric is much less understood, for example thanks to recent work on the resolution of singularities of cotangent sheaves of [BdSBGM17].

It is also worth noticing that the Laplacian formula could be fairly easily extended to the case of a complex surface germ with non isolated singularity by adding a correcting term involving the multiplicities of generic hyperplane sections of $X$ at points of $\mathrm{Sing}X \smallsetminus \{0\}$. These correcting terms would appear in a generalized statement of Proposition 4.20, that is in the computation of the Euler Characteristic of the part $F_v$ of the Milnor fiber of a generic linear form on $(X,0)$. Moreover, the fact that our formula enables us to recover a Lê-Greuel type formula for a particular function suggests the possibility of extending the Laplacian formula to more general settings such as the metric behavior of a pair of holomorphic germs $(f,g)\colon (X,0) \to (\mathbb{C}^2,0)$, which as of now is still poorly understood.

Finally, let us remark that the inner rates were used in [BNP14] to study the Lipschitz classification of the inner metric surface germs $(X,0)$. While on this paper we chose to focus on a metric germ $(X,0)$, and not on its Lipschitz class, our methods seem very well adapted to study questions of bi-Lipschitz geometry. For example, the valuation-theoretic point of view permits to give an elegant valuative version of the complete invariant on the inner Lipschitz geometry of $(X, 0)$ constructed in [BNP14].

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Let us give a short outline of the structure of the paper. All the material about non-archimidean links, dual graphs, and potential theory on metric graphs that we need in the paper is recalled in section 2. Section 3 is devoted to the construction of the inner rate function and to the proof of its basic properties, such as its piecewise linearity. In Section 4 we state and prove our main result, Theorem 4.2, which is a stronger version of Theorem A. Section 5 is devoted to proving a slightly more precise version of Theorem A, Corollary 5.1, and discuss some applications.

We aimed to make the paper entirely self-contained. The main definitions and results are illustrated with the help of a recurring example, that of the $E_8$ surface singularity, which is treated in detail throughout the paper (see Examples 2.4, 2.6, 2.8, 3.7, 4.1, and 4.4).

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## 2. Preliminaries

2.1. **Non-archimedean links.** Throughout the paper, $(X, 0)$ will always be an isolated complex surface singularity. We will begin by introducing the valuation space we will work with, the non-archimedean link of $(X, 0)$.

Denote by $\mathcal{O} = \widehat{\mathcal{O}_{X,0}}$ the complete local ring of $X$ at 0. A (rank 1) *semivaluation* on $\mathcal{O}$ is a map $v \colon \mathcal{O} \to \mathbb{R} \cup \{+\infty\}$ such that

(i) $v(fg) = v(f) + v(g)$ ;
(ii) $v(f + g) \geq \min\{v(f), v(g)\}$.

In particular, it follows from the first condition that $v(0) = +\infty$. However, we do not require 0 to be the only element sent to $+\infty$, which is why our maps are only semivaluation rather than valuations. If $v$ is a semivaluation on $\mathcal{O}$, the valuation of $v$ on the maximal ideal $\mathfrak{M}$ of $\mathcal{O}$ is defined as $v(\mathfrak{M}) = \inf \big\{ v(f) \,\big|\, f \in \mathfrak{M} \big\}$. Observe that by definition this integer can be computed as the valuation of the equation of a generic hyperplane section of $(X, 0)$.

**Example 2.1.** The main example of (semi)valuation that we will consider in this paper is the following. Let $\pi \colon X_\pi \to X$ be a good resolution of $(X, 0)$ and let $E \subset \pi^{-1}(0)$ be a prime divisor. Then the map

$$\begin{aligned} v_E \colon \mathcal{O} &\longrightarrow \mathbb{R}_+ \cup \{+\infty\} \\ f &\longmapsto \frac{\mathrm{ord}_E(f)}{\mathrm{ord}_E(\mathfrak{M})} \end{aligned}$$

is a valuation on $\mathcal{O}$. We call it the *divisorial valuation* associated with $E$. Note that this valuation does not depend on the choice of $\pi$, in the following sense: if $\pi' = \pi \circ \rho$ is another good resolution of $(X,0)$ that dominates $\pi$, then $v_E = v_{E'}$, where $E'$ denotes the strict transform of $E$ in $X_{\pi'}$. If $v$ is the divisorial valuation associated with a prime exceptional divisor of a good resolution $\pi\colon X_\pi \to X$, we will generally denote by $E_v$ this prime divisor, and by $m_v$ the positive integer $\mathrm{ord}_{E_v}(\mathfrak{M})$, which we call the *multiplicity* of $E_v$. Indeed, observe that $m_v$ is also the multiplicity of $E_v$ in $\pi^{-1}(0)$, when the latter is considered with its natural scheme structure. In practice, $m_v$ is usually computed as the order of vanishing along $E_v$ of the pullback to $X_\pi$ of the equation of a generic hyperplane section of $(X,0)$.

We are interested in the set of those semivaluations on $\mathcal{O}$ that are trivial on the constants and that can be normalized by requiring that the valuation of the maximal ideal is 1.

**Definition 2.2.** The non-archimedean link of $(X,0)$ is the topological space whose underlying set is

$$\mathrm{NL}(X,0) = \big\{ v\colon \mathcal{O} \to \mathbb{R}_+ \cup \{+\infty\} \text{ semivaluation} \;\big|\; v(\mathfrak{M}) = 1 \text{ and } v|_{\mathbb{C}} = 0 \big\}$$

and whose topology is induced from the product topology of $\big(\mathbb{R}_+ \cup \{+\infty\}\big)^{\mathfrak{M}}$ (that is, it is the topology of the point-wise convergence).

**Remark 2.3.** The non-archimedean link can be endowed with an additional analytic structure, by considering the space of semi-valuations on $\mathcal{O}$ that are trivial on $\mathbb{C}$ as a non-archimedean analytic space in the sense of Berkovich [Ber90]. This point of view was developed in [Fan18], where it was used to obtain a non-archimedean characterization of the essential valuations of a surface singularities, and later in [FFR18] to give a characterization of sandwiched surface singularities. However, the analytic structure of $\mathrm{NL}(X,0)$ will play no role in this paper.

The non-archimedean link $\mathrm{NL}(X,0)$ is a useful object if one wants to study the resolutions of $(X,0)$, as we will now explain. We call *good resolution* of $(X,0)$ a proper morphism $\pi\colon X_\pi \to X$ such that $X_\pi$ is regular, $\pi$ is an isomorphism outside of its exceptional locus $\pi^{-1}(0)$, and the latter is a strict normal crossing divisor on $X_\pi$. The fact that the exceptional divisor has normal crossing allows us to associate with it its *dual graph* $\Gamma_\pi$, which is the graph whose vertices are in bijection with the irreducible components, and where two vertices of $\Gamma_\pi$ are joined by an edge for each point in the intersection of the corresponding components. We will generally write $v$ for a vertex of $\Gamma_\pi$ and $E_v$ for the associated component of $\pi^{-1}(0)$. We also write $g(v)$ for the genus of the component $E_v$; it will sometimes be useful to think of $g(v)$ as a function with finite support on the topological space underlying $\Gamma_\pi$.

Throughout the paper, most of our resolutions will factor through the blowup of the maximal ideal of $(X,0)$. We call the subset of $\mathrm{NL}(X,0)$ consisting of the divisorial valuations associated with the exceptional components of the blowup of the maximal ideal of $\mathrm{NL}(X,0)$ the set of *$\mathcal{L}$-nodes* of $(X,0)$. In other words, this is the set of the Rees valuations of the maximal ideal of $(X,0)$.

**Example 2.4.** Consider the standard singularity $X = E_8$, which is the hypersurface in $\mathbb{C}^3$ defined by the equation $x^2 + y^3 + z^5 = 0$. A good resolution $\pi\colon X_\pi \to X$ of $(X,0)$ can be obtained by the method described in [Lau71, Chapter II], which consists in considering the projection $\ell = (y,z)\colon (X,0) \to (\mathbb{C}^2,0)$ and, given a suitable resolution $\sigma\colon Y \to \mathbb{C}^2$ of the associated discriminant curve $\Delta\colon y^3 + z^5 = 0$, gives a simple algorithm to compute compute the resolution of $(X,0)$ as a cover of $Y$. The exceptional divisor is a tree of eight $\mathbb{P}^1$ whose dual graph $\Gamma_\pi$ is represented on Figure 1. We label the vertices $v_0, \dots, v_7$, as this will make it simple to refer to this example in the rest of the paper; the negative numbers attached to the $v_i$ denote the self-intersections of the corresponding exceptional components; as no self-intersection is -1 this resolution is minimal. Observe that $\pi$ factors through the blowup of the maximal ideal of $(X,0)$, the only $\mathcal{L}$-node of $(X,0)$ is $v_0$. This example will be detailed throughout the whole paper, see Examples 2.6, 2.8, 3.7, 4.1, and 4.4.

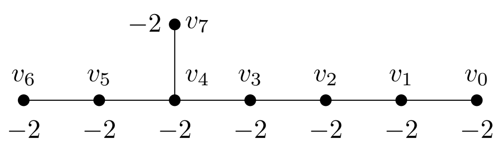



Figure 1.

For any good resolution $\pi\colon X_\pi \to X$ with dual graph $\Gamma_\pi$, there are a natural embedding $i_\pi\colon \Gamma_\pi \to \mathrm{NL}(X,0)$ and a canonical continuous retraction $r_\pi\colon \mathrm{NL}(X,0) \to \Gamma_\pi$ such that $r_\pi \circ i_\pi = \mathrm{Id}_{\Gamma_\pi}$. The embedding $i_\pi$ is defined by sending each vertex $v$ of $\Gamma_\pi$ to the divisorial valuation associated with the component $E_v$, and by sending an edge $e = [v, v']$ that corresponds to a point $p$ of the intersection $E_v \cap E'_v$ to the set of monomial valuations on $X_\pi$ at $p$. These are defined as follows: given $\omega \in [0,1]$ and $f \in \mathcal{O}$, we define a valuation $v_{\pi,p,\omega}$ on $\mathcal{O}$ by setting $v_{\pi,p,\omega}(f) = \inf\big\{(i,j)\cdot(\omega/m_{v'}, (1-\omega)/m_v) \,\big|\, a_{i,j} \neq 0\big\}$, where $f \in \mathcal{O}$, we denote by $\sum a_{i,j}x^i y^j \in \widehat{\mathcal{O}_{X_\pi,P}} \cong \mathbb{C}[[x,y]]$ the image in $\widehat{\mathcal{O}_{X_\pi,P}}$ of the pullback of $f$ through $\pi$, and we chose an isomorphism $\widehat{\mathcal{O}_{X_\pi,P}} \cong \mathbb{C}[[x,y]]$ such that $E_v$ and $E_{v'}$ are defined locally ay $p$ by $x=0$ and $y=0$ respectively. Observe than that $v_{\pi,p,1} = v$ and $v_{\pi,p,0} = v'$. The retraction $r_\pi$ is defined as follows. Given a point $w$ of $\mathrm{NL}(X,0)$, consider its center $\mathrm{c}_\pi(w)$ on $X_\pi$ (we refer to [Vaq00] for the notion of center of a semivaluation). If $\mathrm{c}_\pi(w)$ is a whole component $E_v$ of $\pi^{-1}(0)$ or a point of $E_v$ that is smooth in $\pi^{-1}(0)$, set $r_\pi(w) = v$. If $\mathrm{c}_\pi(w) = p$ lies on the intersection of two components $E_v$ and $E_{v'}$ of $\pi^{-1}(0)$, set $r_\pi(w) = v_{\pi,p,\omega}$, with $\omega = w(x)/\big(w(x)+w(y)\big)$, where as above $x$ and $y$ are local equations for $E_v$ and $E_{v'}$ at $p$.

It is a classical result that the continuous retractions $r_\pi$ induce a homeomorphism from $\mathrm{NL}(X,0)$ to the inverse limit of the dual graphs $\Gamma_\pi$, where $\pi$ ranges through the filtered family of good resolutions of $(X,0)$. This means that the non-archimedean link $\mathrm{NL}(X,0)$ can be thought of as a universal dual graph, making it a very convenient object for studying the combinatorics of the resolutions of $(X,0)$.

Dual resolution graphs can be endowed with a natural metric as follows. Let $\pi\colon X_\pi \to X$ be a good resolution of $(X,0)$ which factors through the blowup of the maximal ideal of $(X,0)$. We define a metric on $\Gamma_\pi$ by declaring the length of an edge $e = [v, v']$ to be

$$\mathrm{length}(e) = \frac{1}{m_v m_{v'}}.$$

**Remark 2.5.** This metric has a natural geometric interpretation, as the opposite $-\operatorname{length}(e) = -\frac{1}{m_v m_{v'}}$ of the length of $e$ coincides with the screw number of the representative of monodromy of $h$ defined as a product of Dehn twists on the annuli defined by any edge between $v$ and $v'$ (see [MMA11, Theorem 7.3.(iv)]). This observation, which will be explained in detail in Section 4.4, will play a key role in the proof of our main result.

**Example 2.6.** Consider again the singularity $(X,0)=(E_8,0)$ of Example 2.4 and its minimal good resolution $\pi\colon X_\pi \to X$. In Figure 2, the vertices of $\Gamma_\pi$ are decorated with the multiplicities of the corresponding exceptional components (in parenthesis), which can be computed by choosing a generic linear form $h\colon (X,0)\to(\mathbb{C},0)$, for example $h=z$. The edges of $\Gamma_\pi$ are decorated with the corresponding lenghts. For example, we can observe that the length of the path from $v_0$ to $v_7$ is $1/6+1/12+1/20+1/30+1/18 = 7/18$. The $\mathcal{L}$-node $v_0$ is decorated with one arrow, representing the fact that the strict transform $h^*$ of $h$ on $X_\pi$ is an irreducible curve passing through the divisor $E_{v_0}$.

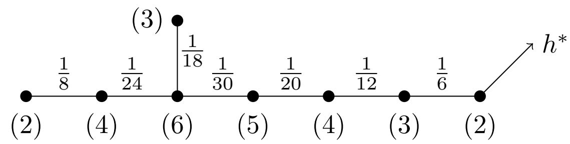



FIGURE 2.

Let $\pi\colon X_\pi \to X$ be a good resolution of $(X,0)$ that factors through the blowup of the maximal ideal of $(X,0)$. Let $p$ is a point of the exceptional divisor $\pi^{-1}$ of $\pi$ that lies on the intersection of two irreducible components $E_v$ and $E_{v'}$ of $\pi^{-1}(0)$ and let $\pi'\colon X_{\pi'} \to X$ be the good resolution of $(X,0)$ obtained from $\pi$ by blowing up $X_\pi$ at the point $p$. Then the exceptional component $E_w$ of $(\pi')^{-1}(0)$ arising from the blowup of $p$ has multiplicity $m_w = m_v + m_{v'}$. Since $1/m_v(m_v+m_{v'}) + 1/m_{v'}(m_v+m_{v'}) = 1/m_v m_{v'}$, this means that the canonical inclusion $\Gamma_\pi \hookrightarrow \Gamma_{\pi'}$, which is in this case bijective, is an isometry. Therefore, by passing to the limit the metrics on the dual graphs $\Gamma_\pi$ define a metric on the non-archimedean link $\mathrm{NL}(X,0)$. We call this metric the *skeletal metric* on $\mathrm{NL}(X,0)$.

2.2. **Generic projections.** We plan to study the surface germ $(X,0)$ using suitable projections to $(\mathbb{C}^2,0)$. We will make use of a classical notion of generic projection due to Teissier.

Fix an embedding of $(X,0)$ in some smooth germ $(\mathbb{C}^n,0)$, and consider the morphism $\ell_\mathcal{D}\colon (X,0)\to(\mathbb{C}^2,0)$ obtained by restricting to $X$ the projection along a $(n-2)$-dimensional linear subspace $\mathcal{D}$ of $\mathbb{C}^n$. Whenever $\ell_\mathcal{D}$ is finite, the associated *polar curve* $\Pi_\mathcal{D}$ is the closure in $(X,0)$ of the ramification locus of $\ell_\mathcal{D}$ in $X\smallsetminus\{0\}$. The Grassmanian variety of $(n-2)$-planes in $\mathbb{C}^n$ contains a dense open set $\Omega$ such that $\ell_\mathcal{D}$ is finite and the family $\{\Pi_\mathcal{D}\}_{\mathcal{D}\in\Omega}$ is well behaved (for example, it is equisingular in a strong sense). We refer the reader to [Tei82, Lemme-clé V 1.2.2] and to [BNP14, Section 3] for a precise definition. We say that a morphisms $\ell\colon (X,0)\to(\mathbb{C}^2,0)$ is a *generic projection* of $(X,0)$ if $\ell=\ell_\mathcal{D}$ for some $\mathcal{D}$ in $\Omega$.

The smallest modification of $(X,0)$ that resolves the base point of the family of polar curves $\{\Pi_\mathcal{D}\}_{\mathcal{D}\in\Omega}$ is the *Nash transform* of $(X,0)$ (see [Spi90, Part III, Theorem

1.2] and [GS82, Section 2]), which is the blowup of $(X,0)$ along its Jacobian ideal. We call the subset of $\mathrm{NL}(X,0)$ consisting of the divisorial valuations associated with the exceptional components of the Nash transform of $(X,0)$ the set of $\mathcal{P}$*-nodes* of $(X,0)$. In another terminology, this is the set of the Rees valuations of the Jacobian ideal of $X$.

A generic projection $\ell\colon (X,0)\to(\mathbb{C}^2,0)$ induces a natural morphism

$$\widetilde{\ell}\colon NL(X,0)\to NL(\mathbb{C}^2,0).$$

Indeed, $\ell$ induces a map $\ell^{\#}\colon \widehat{\mathcal{O}_{X,0}}\to\widehat{\mathcal{O}_{\mathbb{C}^2,0}}$, hence a point of $\mathrm{NL}(X,0)$ (which is a semivaluation on $\widehat{\mathcal{O}_{X,0}}$) gives rise to a point of $\mathrm{NL}(\mathbb{C}^2,0)$ simply by pre-composing the semivaluation with $\ell^{\#}$. A concrete way to compute $\widetilde{\ell}(v)$ for a divisorial valuation $v\in\mathrm{NL}(X,0)$ goes as follows. Take a good resolution $\pi$ of $(X,0)$ such that $v$ is associated with an exceptional component $E_v$ of $\pi^{-1}(0)$ and consider a generic pair of curvettes $\gamma$ and $\gamma'$ of $E_v$. Let $\sigma$ be the minimal sequence of blowups of $(\mathbb{C}^2,0)$ such that the strict transforms of $\ell(\gamma)$ and $\ell(\gamma')$ by $\sigma$ meet the exceptional divisor $\sigma^{-1}(0)$ at distinct points. Then, being generic, they meet $\sigma^{-1}(0)$ at smooth points along the exceptional curve $C_\nu$ created at the last blowup. It can be seen via a standard Hirzebruch–Jung argument that neither the morphism $\sigma$ nor the divisorial valuation $\nu\in NL(\mathbb{C}^2,0)$ depend on the choice of the generic pair $\gamma,\gamma'$, and indeed we have $\widetilde{\ell}(v)=\nu$. Most interestingly, this shows that $\widetilde{\ell}$ does not depend on the choice of the projection $\ell$ as long as it is generic.

2.3. **Laplacians on metric graphs.** We will briefly recall some basic notions of divisor theory on metric graphs.

For us a *graph* is a finite and connected metric graph

$$\Gamma=\big(V(\Gamma),E(\Gamma),l\colon E(\Gamma)\to\mathbb{Q}_{>0}\big),$$

where $V(G)$ is the set of vertices of $\Gamma$, $E(G)$ is the set of its edges, and $l$ attaches a length to each edge of $\Gamma$. We allow $\Gamma$ to have loops and multiple edges. We will freely identify $\Gamma$ with its geometric realization, which is the metric space whose metric is induced by the lengths of its edges.

A *divisor* $D=\sum a_v[v]$ of $\Gamma$ is a finite sum of points of $\Gamma$ with integer coefficients $a_v\in\mathbb{Z}$. We also denote by $D(v)$ the coefficient $a_v$ of a divisor $D$ at a point $v$ of $\Gamma$, and by $\mathrm{Div}(\Gamma)=\bigoplus_{v\in\Gamma}\mathbb{Z}[v]$ the abelian group of divisors of $\Gamma$. The *degree* $\deg(D)$ of $D$ is the integer $\deg(D)=\sum_{v\in\Gamma}D(v)$.

A function $F\colon\Gamma\to\mathbb{R}$ is said to be *piecewise linear* if $F$ is a continuous piecewise affine map with integral slopes (with respect to the metric induced by $l$ on $\Gamma$) and $F$ has only finitely many points of non-linearity on each edge of $\Gamma$.

**Definition 2.7.** If $F\colon\Gamma\to\mathbb{R}$ is a piecewise linear map, its *Laplacian* $\Delta_\Gamma(F)$ is the divisor of $\Gamma$ whose coefficient $\Delta_\Gamma(F)(v)$ at a point $v$ of $\Gamma$ is the sum of the outgoing slopes of $F$ at $v$.

**Example 2.8.** Consider the metric graph $\Gamma_\pi$ associated with the dual graph of the minimal resolution of the singularity $E_8$, as described in Examples 2.4 and 2.6. Let $F$ be the function on $\Gamma_\pi$ which is linear on its edges and such that $\big(F(v_i)\big)_{i=0}^{7}=\big(1,\frac{4}{3},\frac{3}{2},\frac{8}{5},\frac{5}{3},\frac{7}{4},2,2\big)$; we will see in Example 3.7 that this function $F$ is the inner rate of $E_8$. Then it is easy to see that $F$ grows linearly with slope 2 on the path from $v_0$ to $v_6$, while it grows linearly with slope 6 on the edge $[v_4,v_7]$. Therefore, its Laplacian is the divisor $\Delta_{\Gamma_\pi}(F)=2[v_0]+6[v_4]-2[v_6]-6[v_7]$.

Observe that, as we do not require $F$ to be linear inside the edges of $\Gamma$, its Laplacian $\Delta_\Gamma(F)$ is not necessarily supported on $V(\Gamma)$. Moreover, since every segment $[v,v']$ in $\Gamma$ such that $F|_{[v,v']}$ is linear contributes with the same slope but opposite signs to the Laplacian $\Delta_\Gamma(F)(v)$ and $\Delta_\Gamma(F)(v')$ at $v$ and $v'$ respectively, the Laplacian $\Delta_\Gamma(F)$ of $F$ is a divisor of degree 0.

A function $f\colon \Gamma' \to \Gamma$ between two metric graphs induces a natural map $f_*\colon \mathrm{Div}(\Gamma') \to \mathrm{Div}(\Gamma)$ defined by sending a divisor $D = \sum_{v'\in\Gamma'} a_{v'}[v']$ to the divisor $f_*D = \sum_{v\in\Gamma} b_v[v]$, where $b_v = \sum_{v'\in f^{-1}(v)} a_v$.

Similarly, we call *divisor* on $\mathrm{NL}(X,0)$ a finite sum of points of $\mathrm{NL}(X,0)$ with integer coefficients, and we denote by $\mathrm{Div}\big(\mathrm{NL}(X,0)\big) = \bigoplus_{v\in\mathrm{NL}(X,0)} \mathbb{Z}[v]$ the abelian group of divisors on $\mathrm{NL}(X,0)$. Then, if $\pi\colon X_\pi \to (X,0)$ is a good resolution of $(X,0)$, the retraction map $r_\pi\colon \mathrm{NL}(X,0) \to \Gamma_\pi$ induces a map of divisors $(r_\pi)_*\colon \mathrm{Div}\big(\mathrm{NL}(X,0)\big) \to \mathrm{Div}(\Gamma_\pi)$ by the same formula as above.

## 3. The inner rate function

Let $(X,0) \subset (\mathbb{C}^n,0)$ be a surface germ with isolated singularities. In this section we define the inner rate function on a non-archimedean link $\mathrm{NL}(X,0)$ and prove its basic property.

We will use the big-Theta asymptotic notations of Bachmann–Landau in the following form: given two function germs $f,g\colon \big([0,\infty),0\big) \to \big([0,\infty),0\big)$ we say $f$ is *big-Theta* of $g$ and we write $f(t) = \Theta\big(g(t)\big)$ if there exist real numbers $\eta > 0$ and $K > 0$ such that for all $t$ if $f(t) \le \eta$ then $K^{-1}g(t) \le f(t) \le Kg(t)$.

Let $(\gamma,0)$ and $(\gamma',0)$ be two distinct germs of complex curves on the surface germ $(X,0) \subset (\mathbb{C}^n,0)$, and denote by $S_\epsilon$ the sphere in $\mathbb{C}^n$ having center 0 and radius $\epsilon > 0$. Denote by $d_i$ the inner distance on $(X,0)$. The *inner contact* between $\gamma$ and $\gamma'$ is the rational number $q_i = q_i^X(\gamma,\gamma')$ defined by

$$d_i\big(\gamma \cap S_\epsilon, \gamma' \cap S_\epsilon\big) = \Theta(\epsilon^{q_i}).$$

**Remark 3.1.** While the existence of the inner contact $q_i^X(\gamma,\gamma')$ and its rationality can be deduced from the work of [KO97], in the case that interests us they can also be seen as a consequence of the next lemma.

The following lemma is fundamental, as it will allow us to define the inner rate of a divisorial valuation of $\mathrm{NL}(X,0)$.

**Lemma 3.2.** *Let $v \in \mathrm{NL}(X,0)$ be a divisorial valuation on $(X,0)$ and let $\pi\colon X_\pi \to X$ be a good resolution of $(X,0)$ which factors through the blowup of the maximal ideal and through the Nash transform of $(X,0)$ and such that $v$ is the divisorial valuation associated with an exceptional component $E_v$ of $\pi^{-1}(0)$. Consider two curvettes $\gamma^*$ and $\widetilde{\gamma}^*$ of $E_v$ meeting it at distinct points, and write $\gamma = \pi(\gamma^*)$ and $\widetilde{\gamma} = \pi(\widetilde{\gamma}^*)$. Then the inner contact $q_i^X(\gamma,\widetilde{\gamma})$ between $\gamma$ and $\widetilde{\gamma}$ only depends on $v$ and not on the choice of $\pi$, $\gamma^*$, and $\widetilde{\gamma}^*$. Moreover, $m_v q_i^X(\gamma,\widetilde{\gamma})$ is an integer and $q_i^X(\gamma,\widetilde{\gamma}) \ge 1$*

**Definition 3.3.** We denote by $q_v$ the rational number $q_i^X(\gamma,\widetilde{\gamma})$, and call it the *inner rate* of $v$.

**Remark 3.4.** It is worth noticing that the knowledge of all the inner rates $q_v$ allows one to compute the inner contact between any two complex curve germs $(\gamma,0)$ and $(\gamma',0)$ on $(X,0)$. Indeed, assume that the good resolution $\pi\colon X_\pi \to X$ of $(X,0)$

also resolves the complex curve $\gamma \cup \gamma'$ and let $v$ and $v'$ be the vertices of $\Gamma_\pi$ such that $\gamma^* \cap E_v \neq \emptyset$ and ${\gamma'}^* \cap E_{v'} \neq \emptyset$ respectively. Then $q_{inn}(\gamma, \gamma') = q_{v,v'}$, where $q_{v,v'}$ is the maximum, taken over all injective paths $\gamma$ in $\Gamma_\pi$ between $v$ and $v'$, of the minimum of the inner rates of the vertices of $\Gamma_\pi$ contained in $\gamma$. We refer to [NPP18, Proposition 15.3] for details.

*Proof of Lemma 3.2.* In the course of the proof we will use the outer contact between $\gamma$ and $\gamma'$, which is the rational number $q_o = q_o(\gamma, \gamma')$ defined by $d_o\big(\gamma \cap S_\epsilon, \gamma' \cap S_\epsilon\big) = \Theta(\epsilon^{q_o})$, where $d_o$ is the outer distance $d_o(x, y) = ||x - y||_{\mathbb{C}^n}$. It is simple to see that $q_o$ can also be defined by $d_o\big(\gamma \cap \{z = \epsilon\}, \gamma' \cap \{z = \epsilon\}\big) = \Theta(\epsilon^{q_o})$, whenever $z$ is a generic linear form for $(X, 0)$.

First observe that in the smooth case the result comes from classical theory of plane curves singularities. Indeed, in $\mathbb{C}^2$ the inner and outer metrics coincide, and if $C_i$ is an exceptional component of a composition of blowups of points $\sigma\colon Y \to \mathbb{C}^2$ starting with the blowup of the origin, then for every pair of distinct curves $(\delta, 0)$ and $(\delta', 0)$ whose strict transforms by $\sigma$ meet $C_i$ at distinct smooth points of $\sigma^{-1}(0)$, the contact $q_i^{\mathbb{C}^2}(\delta, \delta') = q_o(\delta, \delta')$ coincides with the contact exponent between their Puiseux series (see for example [GBT99, page 401]) and does not depend on the choice of the pair $\delta, \delta'$.

Let us now focus on the general case. Denote by $p$ the point of $E_v$ where $\gamma*$ passes through and consider coordinates $(z_1, \ldots, z_n)$ of $\mathbb{C}^n$ such that $\ell|_X\colon (z_1, \ldots, z_n) \to (z_1, z_2)$ is a generic projection for the surface $(X, 0)$ and such that the strict transform $\Pi^*$ of the polar curve $\Pi$ of $\ell$ by $\pi$ does not pass through $p$, nor do the strict transforms of the curves $X \cap \{z_i = 0\}$ for all $i = 1, \ldots, n$. Choose local coordinates $(u_1, u_2)$ centered at $p$ such that $E_v$ has local equation $u_1 = 0$, $\gamma*$ has local equation $u_2 = 0$, and such that $(z_1 \circ \pi)(u_1, u_2) = u_1^{m_v}$ locally. One can then express the other coordinates in terms of $(u_1, u_2)$ as follows:

$$(z_2 \circ \pi)(u_1, u_2) = u_1^{m_v} f_{2,0}(u_1) + u_1^{m_v q_2} \sum_{j \geq 1} u_2^j f_{2,j}(u_1), \tag{1}$$

and for $i = 3, \ldots, n$,

$$(z_i \circ \pi)(u_1, u_2) = u_1^{m_v} f_{i,0}(u_1) + u_1^{m_v q_i} \sum_{j \geq 1} u_2^j f_{i,j}(u_1),$$

for suitable choices of $f_{i,j}(u_1) \in \mathbb{C}\{\{u_1\}\}$ and $1 \leq q_i \in \mathbb{Q}$. Replacing $z_2$ by a generic combination of the functions $(z_1, \ldots, z_n)$, we can assume that $q_i \geq q_2 \geq 1$ for all $i \geq 3$. We will prove the following claim:

**Claim 1.** For any curvette $\widetilde{\gamma}^*$ of $E_v$ meeting $E_v$ at a point distinct from $p$, we have $q_i^X(\gamma, \widetilde{\gamma}) = q_2$.

Denote by $\gamma_t^*$ the curvette of $E_v$ defined by the equation $u_2 = t$ (so that in particular we have $\gamma_0 = \gamma$), set $\gamma_t = \pi(\gamma_t^*)$, and let $D$ be a disc neighborhood of $p$ in $E_v$ which is contained in a neighborhood on which the local coordinates $(u_1, u_2)$ are defined. Then for every $t \in D \setminus \{0\}$ we have

$$d_o\big(\gamma \cap \{z_1 = \epsilon\}, \gamma_t \cap \{z_1 = \epsilon\}\big) = \Theta(\epsilon^{q_2}).$$

Now, for every $t \in D \setminus \{0\}$, replacing $\gamma_t$ by any other curvette $\widetilde{\gamma}^*$ passing through $(u_1, u_2) = (0, t)$ still gives

$$d_o\big(\gamma \cap \{z_1 = \epsilon\}, \widetilde{\gamma} \cap \{z_1 = \epsilon\}\big) = \Theta(\epsilon^{q_2}),$$

since $\widetilde{\gamma}^*$ is defined by a parametrization of the form $u_2 = t + h.o.$, where $h.o.$ denotes a sum of higher order terms. In particular, we have $q_o(\gamma, \widetilde{\gamma}) = q_2$.

On the other hand, let $\ell\colon (X,0) \to (\mathbb{C}^2, 0)$ be a generic projection for $(X,0)$. By [BNP14, Proposition 3.3], the local bilipschitz constant of the cover $\ell$ is bounded outside $\pi(W)$, where $W$ is any analytic neighborhood of $\Pi^*$ in $X_\pi$. Since $\Pi^*$ does not pass through $p$, we can take $D$ and $W$ small enough that $D \cap W = \emptyset$. Therefore, the local bilipschitz constant of $\ell$ being bounded on $\cup_{t\in D}\gamma_t$, we have $q_i^X(\gamma, \widetilde{\gamma}) = q_i\big(\ell(\gamma), \ell(\widetilde{\gamma})\big)$ as long as $\widetilde{\gamma}^*$ passes through a point of $E_v \cap D$.

Since the coincidence exponent between the curves $\ell(\gamma)$ and $\ell(\widetilde{\gamma})$ in $(\mathbb{C}^2, 0)$ equals $q_2$, we deduce that $q_i^X(\gamma, \widetilde{\gamma}) = q_o\big(\ell(\gamma), \ell(\widetilde{\gamma})\big) = q_2$. This proves that $q_i^X(\gamma, \widetilde{\gamma}) = q_2$ does not depend on the choice of the curvette $\widetilde{\gamma}^*$ providing $\widetilde{\gamma}^* \cap E_v$ is in the neighborhood $D$ of $\gamma^*$.

We now have to prove that $q_2$ does not depend on the point $p$, and that $q_i^X(\gamma, \widetilde{\gamma}) = q_2$ for any pair of curvettes $\gamma^*$ and $\widetilde{\gamma}^*$ meeting $E_v$ at distinct points.

Let us now prove the following Claim:

**Claim 2.** The contact order $q_2$ does not depend on the point $p$.

Let $p'$ be another smooth point of $\pi^{-1}(0)$ on $E_v$, let $(\delta)^*$ be a curvette of $E_v$ through $p'$ and let $\widetilde{\delta}^*$ be a neighbor curvette. Let $\delta$ and $\widetilde{\delta}$ be the images through $\pi$ of $\delta^*$ and $\widetilde{\delta}^*$ respectively. Let $q_2(p')$ be the rate $q_2$ obtained as above by taking local coordinates centered at $p'$ instead of $p$. We can assume without loss of generality that the projection $\ell$ is also generic for these coordinates in the sense above. Then we have $q_o\big(\ell(\delta), \ell(\widetilde{\delta})\big) = q_2(p')$. Consider the minimal sequence of blowups $\sigma\colon Y \to \mathbb{C}^2$ such that one of the irreducible components $C$ is an of $\sigma^{-1}(0)$ corresponds to the valuation $\widetilde{\ell}(v)$ (where $\widetilde{\ell}$ is defined in Section 2.2). Then the strict transforms of the four curves $\gamma$, $\widetilde{\gamma}$, $\delta$ and $\widetilde{\delta}$ intersect $C$ at four distinct points of $C$ which are smooth points of $\sigma^{-1}(0)$. Therefore, we have $q_2 = q_o(\gamma, \widetilde{\gamma}) = q_o(\delta, \widetilde{\delta}) = q_2(p')$. This proves Claim 2.

Let us now take any pair of curvettes $\gamma^*$ and $\widetilde{\gamma}^*$ meeting $E_v$ at distinct points $p$ and $\widetilde{p}$. Since the contact of $\gamma$ wit the $\pi$-image of any neighbor curvette equals $q_2$, we then have $q_i^X(\gamma, \widetilde{\gamma}) \leq q_2$. Moreover, by compactness of $E$, we can choose a finite sequence of smooth points $p_1, p_2, \ldots, p_s$ of $\pi^{-1}(0)$ on $E_v$, such that $p_1 = p$, $p_s = \widetilde{p}$ and for all $i = 1, \ldots, s-1$, $q_i^X(\gamma_i, \gamma_{i+1}) = q_2$ where $\gamma_i^*$ and $\gamma_{i+1}^*$ pass through $p_i$ and $p_{i+1}$ respectively. We then have

$$q_i^X(\gamma, \widetilde{\gamma}) \geq \min_{i=1,\ldots,s-1} \big(q_i^X(\gamma_i, \gamma_{i+1})\big) = q_2.$$

Putting this all together, we deduce that $q_i^X(\gamma, \widetilde{\gamma}) = q_2$.

Finally, observe that $q_i^X(\gamma, \widetilde{\gamma})$ only depends on $v$ and not on $E_v$, since all the computations performed above are unchanged if we first blowup a closed point of $E_v$. ∎

**Remark 3.5.** The second claim in the proof of Lemma 3.2 can also be proved by a computation using local coordinates in the resolution as follows. Let $p'$ be another smooth point of $\pi^{-1}(0)$ on $E_v$. By the genericity of $z_1$ and $z_2$, we can assume that the normal form of equation (1) is also valid at $p'$, that is, there exists a coordinate system $(u', v')$ centered at $p'$ such that:

$$(z_1 \circ \pi)(u'_1, u'_2) = u_1'^{m_v},$$
$$(z_2 \circ \pi)(u'_1, u'_2) = u_1'^{m_v} g_{2,0}(u'_1) + u_1'^{m_v q_v(p')} \sum_{j\geq 1} u_2'^{j} g_{2,j}(u'_1).$$

Observe that that:

$$dz_1 \wedge dz_2 = u_1'^{m_v(1+q_v(p'))-1} \Big( \sum_{j\geq 1} j u_2'^{j-1} g_{2,j}(u'_1) \Big) du_1 \wedge du_2.$$

We can interpret the exponent $m_v\big(1 + q_v(p')\big) - 1$ as the maximal order of the exceptional divisor that factors through $dz_1 \circ \pi \wedge dz_2 \circ \pi$. Since this order is independent of the point $p' \in E_v$, we conclude that $q_v(p) = q_v(p')$ for every point $p' \in E_v$.

**Example 3.6.** Consider the minimal resolution $\sigma\colon Y \to \mathbb{C}^2$ of the plane curve $\Delta\colon y^3 + z^5 = 0$. Let us label the vertices of $\Gamma_\sigma$ as $\nu_0, \nu_1, \nu_2$, and $\nu_3$ in their order of appearance in the resolution process. The inner rates are computed as contact exponent between Puiseux series of neighbor curvettes. This gives the tree $T_0$ of Figure 3, where each vertex is weighted by the self intersection of the corresponding exceptional curve and with its the inner rates (in bold), while the arrow denotes the strict transform of $\Delta$. Note that Lemma 3.8 below will also provide a simple way to compute the inner rates of divisorial valuations on $(X, 0)$.

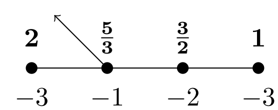



FIGURE 3.

**Example 3.7.** Let us consider again the singularity $X = E_8$ of example 2.4. The projection $\ell = (y, z)\colon (X, 0) \to (\mathbb{C}^2, 0)$ is generic for $(X, 0)$ and its discriminant curve is the curve $\Delta$ introduced in example 3.6. Using Laufer's algorithm in [Lau71], we observe that the $\tilde{\ell}(v_i), i = 0, \ldots, 7$ are exactly the vertices of the dual graph $T$ obtained by blowing-up each intersecting points between two components of $\sigma^{-1}(\Delta)$. The inner rates on $T$ are given on Figure 4. Therefore the inner rate function on the vertices of $\Gamma_\pi$ coincides with the function introduced in Example 2.8.

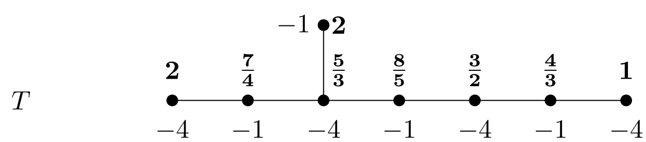



FIGURE 4.

The following result is what allows us to compute in a simple way the inner rate of any divisorial valuation of $(\mathbb{C}^2, 0)$, and more generally that of any divisorial valuation of a singular germ $(X, 0)$ if we know the inner rates of the vertices of the dual graph $\Gamma_\pi$ of a suitable good resolution $\pi$ of $(X, 0)$.

**Lemma 3.8.** *Let $\pi\colon X_\pi \to X$ be a good resolution of $(X,0)$ that factors through the blowup of the maximal ideal and through the Nash transform of $(X,0)$. Let $p$ be a point of the exceptional divisor $\pi^{-1}(0)$ of $\pi$ and let $E_w$ be the exceptional component created by the blowup of $X_\pi$ at $p$. Then:*

*(i) If $p$ is a smooth point of $\pi^{-1}(0)$ and $E_v$ is the irreducible component of $\pi^{-1}(0)$ on which $p$ lies, then*

$$m_w = m_v \quad \text{and} \quad q_w = q_v + \frac{1}{m_v}$$

*(ii) If $p$ lies on the intersection of two irreducible components $E_v$ and $E_{v'}$ of $\pi^{-1}(0)$, then*

$$m_w = m_v + m_{v'} \quad \text{and} \quad q_w = \frac{q_v m_v + q_{v'} m_{v'}}{m_v + m_{v'}}.$$

*Proof.* Assume first that $p$ is a smooth point of $\pi^{-1}(0)$. We use again the notations of the proof of Lemma 3.2: in local coordinates $(u_1,u_2)$ centered at $p$, we have $(z_1\circ\pi)(u_1,u_2) = u_1^{m_v}$ and

$$(z_2\circ\pi)(u_1,u_2) = u_1^{m_v} f_{2,0}(u_1) + u_1^{m_v q_2} \sum_{j\geq 1} u_2^j f_{2,j}(u_1).$$

Let us prove that $f_{2,1}(0)\neq 0$. Let $\Pi$ be the polar curve of $\ell$. Its total transform $\pi^{-1}(\Pi)$ by $\pi$ is the critical locus of $\ell\circ\pi$. The Jacobian matrix of $\ell\circ\pi$ is:

$$\begin{pmatrix} m_v u_1^{m_v-1} & 0 \\ * & u_1^{m_v q_2}\big(f_{2,1}(u_1) + 2u_2 f_{2,2}(u_1) + \cdots\big) \end{pmatrix}$$

thus $\pi^{-1}(\Pi)$ has equation $m_v u_1^{m_v + m_v q_2 - 1}\big(f_{2,1}(u_1) + 2u_2 f_{2,2}(u_1) + \cdots\big) = 0$ and the strict transform $\Pi^*$ has equation $f_{2,1}(u_1) + 2u_2 f_{2,2}(u_1) + \cdots = 0$. Since $\Pi^*$ does not pass through $p$, this implies $f_{2,1}(0)\neq 0$.

Let $e_p$ be the blowup of $p$. In the coordinates chart $(u_1',u_2')\mapsto(u_1',u_2'u_1')$, we have: $(z_1\circ\pi\circ e_p)(u_1',u_2') = {u_1'}^{m_v}$ and

$$(z_2\circ\pi\circ e_p)(u_1',u_2') = {u_1'}^{m_v} f_{2,0}(u_1') + {u_1'}^{m_v q_2 + 1} \sum_{j\geq 1} {u_2'}^j {u_1'}^{j-1} f_{2,j}(u_1').$$

Therefore $m_w = m_v$. Fixing $u_2'\neq 0$ and comparing with the equation (1), we obtain $m_w q_w = m_v q_2 + 1$ by using the Claim of the proof of Lemma 3.2. This proves (i).

Assume now that $p$ is an intersecting point between two irreducible components $E_v$ and $E_{v'}$. In local coordinates $(u_1,u_2)$ centered at $p$, we can assume without loss of generality that $(z_1\circ\pi)(u_1,u_2) = u_1^{m_v}u_2^{m_{v'}}$ and $(z_i\circ\pi)(u_1,u_2)\in\mathbb{C}\{u_1,u_2\}$; in particular, we consider their Taylor expansion

$$(z_i\circ\pi)(u_1,u_2) = \sum_{\alpha\in\mathbb{N}^2} T_{i\alpha} u_1^{\alpha_1} u_2^{\alpha_2}$$

Now, let $\Lambda = \big\{\alpha\in\mathbb{N}^2 \,\big|\, \exists q\in\mathbb{Q} \text{ such that } q\cdot\alpha = (m_v, m_{v'})\big\}$. Since $(z_i\circ\pi)(u_1,u_2)$ are convergent power series, a sub-series is also convergent, and therefore

$$\widetilde{g}_i(u_1,u_2) = \sum_{\alpha\in\Lambda} T_{i\alpha} u_1^{\alpha_1} u_2^{\alpha_2}$$

is an analytic function. Furthermore, we know that $u_1^{m_v}u_2^{m_{v'}}$ divides $(z_i\circ\pi)(u_1,u_2)$, so $\widetilde{g}_i(u_1,u_2) = u_1^{m_v}u_2^{m_{v'}} g_i(u_1,u_2)$ and:

$$(z_i\circ\pi)(u_1,u_2) = u_1^{m_v}u_2^{m_{v'}} g_i(u_1,u_2) + u_1^{b_v}u_2^{b_{v'}} h_i(u_1,u_2)$$

where $b_v \geq m_v$, $b_{v'} \geq m_{v'}$ and, without loss of generality, $h_2(u_1, u_2)$ is not identically zero over $E_v$ and $E_{v'}$. Now consider a point $(0, a_2)$ (the same computation works for $(a_1, 0)$) and let us compare this Taylor expansion with the normal form given in equation (1). Indeed, consider the analytic change of coordinates:

$$u = u_1(u_2 - a_2)^{\frac{m_{v'}}{m_v}}, \quad v = u_2$$

which is centred at $(0, a_2)$ and note that $(z_1 \circ \pi)(u_1, u_2) = u^{m_v}$. Furthermore, it follows from direct computation and the definition of $\Lambda$ that:

$$u_1^{m_v} u_2^{m_{v'}} g_i(u_1, u_2) = u^{m_v} g_i(u)$$

so, these terms contribute only to the terms $f_{i0}$ of the normal form. Furthermore, if $\alpha \in \mathbb{N} \setminus \Lambda$, then:

$$u_1^{\alpha_1} u_2^{\alpha_2} = u^{\alpha_1} (v - a_2)^{\frac{\alpha_2 m_{v'} - \alpha_1 m_v}{m_{v'}}} = u^{\alpha_1} U_\alpha(u, v)$$

where $U_\alpha(u, v)$ is a non-constant unit whose derivative in respect to $v$ is non-zero. By comparing the normal forms, it follows, that $b_v = m_v q_v$ (and by the analogous argument, that $b_{v'} = m_{v'} q_{v'}$), which yields:

$$(z_i \circ \pi)(u_1, u_2) = u_1^{m_v} u_2^{m_{v'}} g_i(u_1, u_2) + u_1^{m_v q_v} u_2^{m_{v'} q_{v'}} h_i(u_1, u_2)$$

Now, since $\pi$ factors through the Nash transform of $(X, 0)$, we can suppose without loss of generality that the polar curve in respect to $(z_1, z_2)$ does no pass through $p$. This implies that the following two form has suppose in the exceptional divisor:

$$d(z_1 \circ \pi) \wedge d(z_2 \circ \pi) = d(u_1^{m_v} u_2^{m_{v'}}) \wedge d(u_1^{q_v m_v} u_2^{q_{v'} m_{v'}} h_2(u_1, u_2))$$

Denoting by $\partial = m_{v'} u_1 \partial_{u_1} - m_v u_2 \partial_{u_2}$, we get that:

$$d(z_1 \circ \pi) \wedge d(z_2 \circ \pi) = u_1^{m_v - 1} u_2^{m_{v'} - 1} \partial(u_1^{q_v m_v} u_2^{q_{v'} m_{v'}} h_2(u_1, u_2)) du_1 \wedge du_2$$

this implies that $h_2$ must be a unit at $p$. Then part (ii) of the Lemma follows from a simple direct computation. □

The starting point of our study of inner rates via potential theory on dual graphs is the following result, which states that the inner rates extend to a continuous, and that this function is piecewise-linear with respect to the metric defined in Section 2.3.

**Lemma 3.9.** *There exists a unique continuous function*

$$\mathcal{I}_X \colon NL(X, 0) \to \mathbb{R}_{\geq 1} \cup \{\infty\}$$

*such that $\mathcal{I}_X(v) = q_v$ for every divisorial point $v$ of $NL(X, 0)$. If $\pi$ is a good resolution of $(X, 0)$ that factors through the blowup of the maximal ideal and the Nash transform of $(X, 0)$, then $\mathcal{I}_X$ is linear on the edges of $\Gamma_\pi$ with integer slopes. Moreover, $\mathcal{I}_X$ satisfies $\mathcal{I}_X = \mathcal{I}_{\mathbb{C}^2} \circ \widetilde{\ell}$.*

*Proof.* Let $\pi$ be as in the statement. We only need to show that the inner rates extend uniquely to a continuous map on $\Gamma_\pi$ which is linear on its edges with integer slopes, as the first part of the statement will then follow immediately from the description of $\mathrm{NL}(X, 0)$ as inverse limit of dual graphs. The fact that the slopes are integer can be verified directly, as on an edge $e = [v, v']$ the slope is $(q_{v'} - q_v)/\operatorname{length}(e) = (q_{v'} - q_v) m_v m_{v'}$, which is an integer by Lemma 3.2. To prove the linearity on the edges, since the subset of $\Gamma_\pi$ consisting of the divisorial points is dense in $\Gamma_\pi$, it is enough to show that the inner rates are linear on this set. Let $e = [v, v']$ be an edge of $\Gamma_\pi$ corresponding to an intersection point $p$ of two components $E_v$ and $E_{v'}$ of $\pi^{-1}(0)$. Since any divisorial point of $e$ is associated with a divisor appearing after a finite composition of point blowups centered over $p$, it is sufficient to prove that $\mathcal{I}_X$ is linear on the set $\{v, v'', v'\}$, where $v''$ is the divisorial valuation associated with the exceptional divisor $E_{v''}$ of the blowup of $X_\pi$ at $p$. Therefore, all we have to show is that

$$\frac{\mathcal{I}_X(v'') - \mathcal{I}_X(v)}{\mathrm{length}([v, v''])} = \frac{\mathcal{I}_X(v') - \mathcal{I}_X(v)}{\mathrm{length}([v, v'])},$$

which follows from the definition of the lengths and from Lemma 3.8.(ii). The fact that $\mathcal{I}_X = \mathcal{I}_{\mathbb{C}^2} \circ \widetilde{\ell}$ follows immediately from hte description of $\widetilde{\ell}$ given in Section 2.2 and from the computation of the inner rates of Lemma 3.2. □

**Remark 3.10.** With any continuous map $F\colon \mathrm{NL}(X,0) \to \mathbb{R}_+ \cup \{+\infty\}$ one can naturally associate an ultrametric (that is, non-archimedean) distance on $\mathrm{NL}(X,0)$ via a standard minimax procedure: the distance between two points $v$ and $v'$ is set to be $e^{-F_{v,v'}}$, where $F_{v,v'}$ is the maximum, taken over all injective paths $\gamma$ in $\mathrm{NL}(X,0)$ between $v$ and $v'$, of the minimum of $F$ on $\gamma$. The observation contained in Remark 3.4 allows us to give a natural geometric interpretation to the ultrametric distance associated with the inner rate function on $\mathrm{NL}(X,0)$.

## 4. The Laplacian of the inner rate

In this section we prove our main result, Theorem 4.2, which computes the Laplacian of the restriction of the inner rate function to the dual graph of any good resolution of $(X,0)$. In order to state it precisely we need to collect a few more definitions.

Let $\pi\colon X_\pi \to (X,0)$ be a good resolution and consider the map $g\colon \Gamma_\pi \to \mathbb{Z}_{\geq 0}$ sending a vertex of $\Gamma_\pi$ to the (arithmetic) genus of the associated divisor and everything else to zero. We then define the *canonical divisor* $K_{\Gamma_\pi}$ of the graph $\Gamma_\pi$ as the divisor $K_{\Gamma_\pi} = \sum_{v\in\Gamma_\pi} m_v \big(\mathrm{val}_{\Gamma_\pi}(v) + 2g(v) - 2\big)[v]$ of $\Gamma_\pi$, where $\mathrm{val}_{\Gamma_\pi}(v)$ denotes the valency of $\Gamma_\pi$ at $v$ (that is the number of edges of $\Gamma_\pi$ adjacent to $e$). Observe that this is indeed an element of $\mathrm{Div}(\Gamma_\pi)$ because for every point $v$ of $\Gamma_\pi$ that is not a vertex we have $\mathrm{val}(v) = 2$ and $g(v) = 0$.

In particular, the canonical divisors $K_{\Gamma_\pi}$ will account for the fact that, since the inner rate grows linearly after blowing up a smooth point on an exceptional component of a good resolution, the Laplacian of $\mathcal{I}|_{\Gamma_\pi}$ at a point $v$ does depend on the choice of a resolution $\pi$ such that $v \in \Gamma_\pi$, and more precisely on the valency $\mathrm{val}_{\Gamma_\pi}(v)$. For this reason, the Laplacians $\Delta_{\Gamma_\pi}(\mathcal{I}|_{\Gamma_\pi})$ will not define a divisor on $\mathrm{NL}(X,0)$ by a limit procedure.

Finally, we need to introduce two divisors on $\mathrm{NL}(X,0)$. Define $L = \sum m_v l_v [v]$, where $v$ ranges over the set of divisorial valuations of $(X,0)$ associated with the exceptional components of the blowup of the maximal ideal of $(X,0)$, and $l_v$ is the number of components of a generic hyperplane section of $(X,0)$ whose strict transforms by the blowup intersect the divisor $E_v$ associated with $v$. Similarly, set $P = \sum m_v p_v [v]$, for $v$ ranging over the set of divisorial valuations of $(X,0)$ associated with the Nash transform of $(X,0)$, where $p_v$ is the number of components of the polar curve of a generic projection of $(X,0)$ whose strict transforms by the Nash transform intersect $E_v$. Observe that by definition the divisor $L$ is supported on the set of $\mathcal{L}$-nodes of $(X,0)$, while $P$ is supported on the set of its $\mathcal{P}$-nodes.

**Example 4.1.** Consider again the singularity $E_8$, whose resolution data has been described in Examples 2.4, 2.6, 2.8, and 3.7. Then the canonical divisor of $\Gamma_\pi$ is $K_{\Gamma_\pi} = -2[v_0] + 6[v_4] - 2[v_6] - 3[v_7]$. Moreover, as can be seen from the discussion of Example 2.6, we have $L = 2[v_0]$. Observe that the divisor $P$ is not supported on $\Gamma_\pi$, since the resolution $\pi$, while a resolution of the polar curve of a generic projection, does not factor through the Nash transform (see [BNP14, Example 3.5] for details). One can verify that $\pi$ factors through the Nash transform after blowing up a suitable smooth point of $E_{v_7}$ and then a suitable smooth point of the resulting exceptional divisor. In particular, we deduce that $(r_\pi)_*P = 3[v_7]$.

4.1. **Statement of the main theorem.** We have now collected all the ingredients needed to state our main theorem in full generality.

**Theorem 4.2** (Laplacian of the inner rate function)**.** *Let* $\pi\colon X_\pi \to (X,0)$ *be a good resolution that factors through the blowup of the maximal ideal of* $(X,0)$*. Then the following equality*

$$\Delta_{\Gamma_\pi}\big(\mathcal{I}_X|_{\Gamma_\pi}\big) = K_{\Gamma_\pi} + 2L - (r_\pi)_*P$$

*holds in* $\mathrm{Div}(\Gamma_\pi)$*.*

**Remark 4.3.** In particular, if $\pi$ is a good resolution of $(X,0)$ which factors through the blowup of the maximal ideal of $X$ at 0 and through the Nash transform of $(X,0)$, then we obtain the statement given in the introduction. Namely, the Laplacian of the inner rate function on the dual graph $\Gamma_\pi$ at a vertex $v$ is precisely $m_v\big(+2l_v - p_v - \chi(\check{E}_v)\big)$, where $l_v$ and $p_v$ are defined as above.

**Example 4.4.** The formula of the theorem can be readily verified in the case of the singularity $E_8$ by combining Example 2.8 and Example 4.1. For example, on the vertex $v_4$, we indeed get $\Delta_{\Gamma_\pi}(\mathcal{I}_X)(v_4) = 6 = K_{\Gamma_\pi}(v_4) = K_{\Gamma_\pi}(v_4) + 2L(v_4) - P(v_4)$.

We introduce another simple combinatorial definition. Let $\Gamma_\pi$ be the dual graph of a good resolution of $(X,0)$ that factors through the blowup of the maximal ideal and through the Nash transform of $(X,0)$. We call *string* of $\Gamma_\pi$ a segment $\mathcal{S}$ in $\Gamma_\pi$ that starts and ends at a vertices $v$ and $v'$ of $\Gamma_\pi$ respectively and such that $\mathcal{S} \smallsetminus \{v,v'\}$ contains no $\mathcal{L}$-nodes, no $\mathcal{P}$-nodes, and no points whose genus is strictly positive or whose valency in $\Gamma_\pi$ is at least 3. Observe that every edge of $\Gamma_\pi$ is a string. The following result, while a straightforward consequence of Theorem 4.2, is of independent interest.

**Corollary 4.5.** *Let* $\pi\colon X_\pi \to (X,0)$ *be a good resolution that factors through the blowup of the maximal ideal and through the Nash transform of* $(X,0)$*. Then, if* $\mathcal{S}$ *is a string of* $\Gamma_\pi$*, the inner rate function* $\mathcal{I}_X$ *is linear on* $\mathcal{S}$*.*

The remaining part of the section will be devoted to the proof of Theorem 4.2. An outline of the proof's method can be found in the introduction.

4.2. **Reduction to dominant resolution and the smooth case.** We will now prove the formula of our main theorem in the case of a smooth surface germ. This is the content of Proposition 4.7.

In order to do so, we will first establish a simple combinatorial lemma that will be used several times throughout the proof of Theorem 4.2.

**Lemma 4.6.** *Let $\pi\colon X_\pi \to X$ and $\pi'\colon X_{\pi'} \to X$ be two good resolution of $(X,0)$ and assume that $\pi'$ factors through $\pi$ which in turn factors through the blowup of the maximal ideal of $(X,0)$. If*

$$\Delta_{\Gamma_{\pi'}}(\mathcal{I}_X|_{\Gamma_{\pi'}}) = K_{\Gamma_{\pi'}} + 2L - (r_{\pi'})_*P$$

*holds, then*

$$\Delta_{\Gamma_{\pi}}(\mathcal{I}_X|_{\Gamma_{\pi}}) = K_{\Gamma_{\pi}} + 2L - (r_{\pi})_*P$$

*holds as well. Moreover, the converse implication is also true if $\pi$ also factors through the the Nash transform of $(X,0)$.*

*Proof.* To unburden the notation, we will write $\Gamma$ and $\Gamma'$ for $\Gamma_\pi$ and $\Gamma_{\pi'}$ respectively. Since $\pi'$ dominates $\pi$, the associated dual graph $\Gamma'$ contains $\Gamma$. Denote by $\tilde{r}\colon \Gamma' \to \Gamma$ the restriction of $r_\pi$ to $\Gamma'$, so that $r_\pi = \tilde{r} \circ r_{\pi'}$. By applying $\tilde{r}_*$ to the first equation of the statement we obtain

$$\tilde{r}_*\Delta_{\Gamma'}(\mathcal{I}_X|_{\Gamma'}) = \tilde{r}_*K_{\Gamma'} + \tilde{r}_*2L - \tilde{r}_* \circ (r_{\pi'})_*P.$$

We will first prove that the left-hand side is equal to $\Delta_\Gamma(\mathcal{I}_X|_\Gamma)$ and that $r_*(K_{\Gamma'}) = K_\Gamma$. As $\pi'$ is obtained by composing the resolution $\pi$ with a finite sequence of point blowups, by induction we can assume without loss of generality that $\pi'$ is obtained by blowing up a single point $P$ of $\pi^{-1}(0)$. If $P$ lies on the intersection of two components of $\pi^{-1}(0)$ then topologically $\Gamma' = \Gamma$, the map $\tilde{r}$ is the identity, and $K_{\pi'} = K_\pi$, so in this case there is nothing to prove. We can therefore assume that $\Gamma'$ is obtained from $\Gamma$ by adding a single vertex $v'$ (corresponding to the exceptional component of the blowup of $P$) and precisely one edge $e$ connecting $v'$ to a vertex $v$ of $\Gamma$. Then we have

$$\Delta_{\Gamma'}(\mathcal{I}_X|_{\Gamma'})(v) = \Delta_\Gamma(\mathcal{I}_X|_\Gamma)(v) + p,$$

where $p$ is the slope of $\mathcal{I}_X$ on the edge $e$ going from $v$ to $v'$. Since the Laplacian of the restriction of $\mathcal{I}_X$ to the edge $e = \tilde{r}^{-1}(v)$ has degree 0, we have

$$p = -\sum_{v'' \in \tilde{r}^{-1}(v) \smallsetminus \{v\}} \Delta_{\Gamma'}(\mathcal{I}_X|_{\Gamma'})(v'').$$

This implies that

$$\Delta_\Gamma(\mathcal{I}_X|_{\Gamma'})(v) = \Delta_{\Gamma'}(\mathcal{I}_X|_\Gamma)(v) + \Delta_{\Gamma'}(\mathcal{I}_X|_\Gamma)(v') = \sum_{v'' \in \tilde{r}^{-1}(v)} \Delta_{\Gamma'}(\mathcal{I}_X|_{\Gamma'})(v''),$$

which is what we wanted to prove. Moreover, observe that $\tilde{r}_*K_{\Gamma'}(v) = K_{\Gamma'}(v) + K_{\Gamma'}(v') = K_{\Gamma'}(v) - m_{v'} = K_{\Gamma'}(v) - m_v$, which is precisely $K_\Gamma(v)$ since the valency of $\Gamma$ in $v$ is equal to the valency of $v$ in $\Gamma'$ minus one. This shows that $\tilde{r}_*$ respects the canonical divisors. Since $\tilde{r}_* \circ (r_\pi)_* = (\tilde{r} \circ r_{\pi'})_* = (r_\pi)_*$, this proves the first part of the lemma. To establish the second part of the lemma, observe that, under the hypothesis that $\pi$ factors through the blowup of the maximal ideal and though the Nash transform of $(X,0)$, both $L$ and $P$ are divisors on $\Gamma_\pi$ and therefore they are stable under $(r_\pi)_*$ and $(r_{\pi'})_*$, and moreover the slope $p$ on the edge $[v,v']$ is equal to $m_v$ (and therefore to $m_{v'}$) thanks to Lemma 3.8.(i). This means that

$$\begin{aligned}\Delta_{\Gamma'}(\mathcal{I}_X|_{\Gamma'})(v) - \Delta_\Gamma(\mathcal{I}_X|_\Gamma)(v) &= m_v = K_{\pi'} - K_\pi \\ &= \big(K_{\pi'} + 2L - (r_{\pi'})_*P\big) - \big(K_\pi + 2L - (r_\pi)_*P\big),\end{aligned}$$

which concludes the proof. ☐

In the smooth case the non-archimedean link $\mathrm{NL}(\mathbb{C}^2,0)$ is a tree, it has divisors $L=[\mathrm{ord}_0]$, the only $\mathcal{L}$-node being the divisorial valuation $\mathrm{ord}_0$ associated with the blowup of $\mathbb{C}^2$ at 0, and $P=0$, as a generic projection to $\mathbb{C}^2$ is unramified and thus there are no $\mathcal{P}$-nodes. The Laplacian of the inner rate function in this very special case is simple to compute thanks to the previous lemma.

**Proposition 4.7.** *Let $\pi\colon X_\pi\to\mathbb{C}^2$ be a sequence of point blowups of $(C^2,0)$ that is not the identity. Then the Laplacian of the inner rate function $\mathcal{I}_{\mathbb{C}^2}$ on the dual graph $\Gamma_\pi$ associated with $\pi$ is*

$$\Delta_{\Gamma_\pi}(\mathcal{I}_{\mathbb{C}^2}|_{\Gamma_\pi}) = K_{\Gamma_\pi} + 2[\mathrm{ord}_o].$$

*Proof.* We will argue by induction on the number of blowups in $\pi$. If $\pi$ is obtained by blowing up the origin of $\mathbb{C}^2$ once, then the associated dual graph consists of the divisorial valuation $\mathrm{ord}_0$ and the formula is immediate, as $\Delta_{\{\mathrm{ord}_0\}}(\mathcal{I})=0$ and $K_{\{\mathrm{ord}_0\}}(\mathrm{ord}_0)=-2$. The inductive step is a direct consequence of the second part of Lemma 4.6. □

We state as a separate result the smooth case of Corollary 4.5, which we obtain as an immediate consequence of Proposition 4.7, since we will need to refer to it in the course of our proof of Theorem 4.2.

**Corollary 4.8.** *Let $\pi\colon X_\pi\to\mathbb{C}^2$ be a sequence of point blowups of $(C^2,0)$ that is not the identity. Then, if $\mathcal{S}$ is a string of $\Gamma_\pi$, the inner rate function $\mathcal{I}_{\mathbb{C}^2}$ is linear on $\mathcal{S}$.*

4.3. **Dehn twists and screw numbers.** Let $f\colon A\to A$ be an orientation preserving diffeomorphism of a disjoint union $A=\coprod_{i=1}^r A_i$ of annuli $A_i\cong S^1\times[0,1]$ which cyclically exchanges the annuli $A_i$ and which is periodic on the union of their boundaries. Let $N$ be an integer such that $f^N$ is the identity on the boundary $\partial A=\coprod_{i=1}^r\partial A_i$. Then, up to isotopy fixed on the boundary $\partial A$, the map $f$ is characterized by a rational number $\tau$ defined as follows. Let us choose an annulus $A_{i_0}$ among the $A_i$ and fix an isomorphism $A_{i_0}\cong S^1\times[0,1]$. Observe that the restriction of $f^N$ to $A_{i_0}$ is, up to isotopy fixed on $\partial A_{i_0}$, a product of Dehn twists. Consider the transversal oriented path $\delta=\{1\}\times[0,1]$ inside $S^1\times[0,1]$, and let us orient the circle $c=S^1\times\{1\}$ in such a way that the intersection number $\delta.c$ of $\delta$ with $c$ on the oriented surface $S^1\times[0,1]$ is equal to $+1$.

**Definition 4.9.** The *screw number* $\tau$ of $f$ is the rational number defined by the following equality in $H_1(S^1\times[0,1],\mathbb{Z})$:

$$N\tau c = f^N(\delta)-\delta.$$

Observe that $\tau$ does not depend on the choice of the integer $N$ such that $f^N$ is the identity on $\partial A$, nor on the choice of the annulus $A_{i_0}$.

In the sequel, we will use the two following simple lemmas.

**Lemma 4.10.** *Set $A=S^1\times[0,1]$ and decompose the annulus $A$ as the union of the two concentric annuli $A_1=S^1\times[0,\frac{1}{2}]$ and $A_1=S^1\times[\frac{1}{2},1]$. Let $f\colon A\to A$ be an orientation preserving diffeomorphism such that $f(A_1)=A_1$, $f(A_2)=A_2$, and there exists $N>0$ such that $f^N$ is the identity on $S^1\times\{0,\frac{1}{2},1\}$. Then, if $f|_{A_1}$ and $f|_{A_2}$ have screw numbers $\tau_1$ and $\tau_2$ respectively, $f$ has screw number $\tau_1+\tau_2$.*

*Proof.* Set $\delta_1 = \{1\} \times [0, \frac{1}{2}]$, $\delta_2 = \{1\} \times [\frac{1}{2}, 1]$ and $\delta = \delta_1 \cup \delta_2 = \{1\} \times [0, 1]$. Set also $c_1 = S^1 \times \{\frac{1}{2}\}$ and $c_2 = S^1 \times \{1\}$, oriented so that $\delta.c_i = +1$ in $A$, so $c_1$ and $c_2$ have the same homology class on $A$. Then $N\tau_1 c_1 = f^N(\delta_1) - \delta_1$ and $N\tau_2 c_2 = f^N(\delta_2) - \delta_2$ hold as equalities of cycles in $(A_1, \partial A_1)$ and $(A_2, \partial A_2)$ respectively. Since $c_1 \cong c_2$ in $A$ and $\delta = \delta_1 \cup \delta_2$, we deduce that $N(\tau_1 + \tau_2)c_1 = f^N(\delta) - \delta$ as cycles in $(A, \partial A)$. This proves that the screw number of $f$ equals $\tau_1 + \tau_2$. □

**Lemma 4.11.** *Let $A = [0, 1] \times S^1$ be an annulus and let $\ell\colon A \to A$ be a cyclic cover of degree* $\deg(\ell)$. *Let $\phi\colon A \to A$ and $\phi'\colon A \to A$ be two orientation preserving diffeomorphisms such that $\ell \circ \phi = \phi' \circ \ell$ and such that both $\phi$ and $\phi'$ have a power which is the identity on the boundary of $A$, and let $\tau$ and $\tau'$ denote the screw numbers of $\phi$ and $\phi'$ respectively. Then $\tau' = \deg(\ell)\tau$.*

*Proof.* Let $N$ be a positive integer such that both $\phi^N$ and $(\phi')^N$ are the identity on $\partial A$. Then, using the same notations as before, the screw number $\tau$ of $\phi$ is defined by $\phi^N \delta - \delta = N\tau c$. Therefore $\ell(\phi^N \delta) - \ell(\delta) = N\tau\ell(c)$. Observe that we have $\ell(\delta) - \delta = 0$ and $\ell(c) - \deg(\ell)c = 0$ in $H_1(A, \mathbb{Z})$. Since $\ell \circ \phi^N = (\phi')^N \circ \delta$, this implies that $(\phi')^N \delta - \delta = N\tau \deg(\ell)c$, proving that the screw number $\tau'$ of $\phi'$ is equal to $\deg(\ell)\tau$. □

4.4. **Skeletal metric and the monodromy of the Milnor fibration.** Let $(X, 0)$ be an isolated surface germ and let $\pi\colon X_\pi \to X$ be a good resolution of $(X, 0)$ which factors through the blowup of the maximal ideal. We will now describe the lengths of the edges of $\Gamma_\pi$ as screw numbers of the monodromy of some pieces of the Milnor fibration of a generic linear form on $(X, 0)$.

Let $B_\epsilon$ be a Milnor ball for $(X, 0)$. Set $D_\eta = \{z \in \mathbb{C} \mid |z| \leq \eta\}$ and $D^*_\eta = D_\eta \smallsetminus \{0\}$. When $0 < \eta << \epsilon$, we call the intersection $T_{\epsilon,\eta} = B_\epsilon \cap L^{-1}(D^*_\eta)$ a *Milnor tube* for $L$. The *Milnor–Lê fibration* of $L$ is the locally trivial fibration $L|_{T_{\epsilon,\eta}}\colon T_{\epsilon,\eta} \to D^*_\eta$ restriction of $L$ to the Milnor tube $T_{\epsilon,\eta}$. The monodromy of $L|_{T_{\epsilon,\eta}}$ admits a quasi-periodic representative $\phi$ which can be defined using the resolution $\pi$ as follows. Consider a fiber $F_t = (L|_{T_{\epsilon,\eta}})^{-1}(t)$. We identify $F_t$ with its inverse image by $\pi$ (recall that $\pi$ is a diffeomorphism outside 0).

For each component $E_v$ of $E = \pi^{-1}(0)$, let $N(E_v)$ be a neighborhood of $E_v$ in $X_\pi$, which is the total space of a normal disc-bundle to $E_v$ in $X_\pi$. Then $\pi^{-1}(B_\epsilon)$ can be identified with a neighborhood of $E$ obtained by plumbing the disc-bundles $N(E_v)$. For each vertex $v$ of $\Gamma_\pi$, we set

$$\mathcal{N}(v) = \overline{N(E_v) \smallsetminus \bigcup_{v' \neq v} N(E_{v'})}.$$

Let $L\colon (X, 0) \to (\mathbb{C}^0)$ be the restriction to $(X, 0)$ os a generic linear form $\mathbb{C}^n \to \mathbb{C}$. For each irreducible component $\gamma$ of $L^{-1}(0)$ in $(X, 0)$, denote by $\gamma*$ the strict transform of $\gamma$ in $X_\pi$, let $D_\gamma$ be a small disc in $E_v \smallsetminus \bigcup_{v' \neq v} E_{v'}$ centered at $\gamma^* \cap E_v$, and let $N(\gamma^*)$ be the restriction of $N(E_v)$ over $D_\gamma$, so that $N(\gamma^*)$ is a small polydisc neighborhood of $\gamma^*$ in $X_\pi$. Set

$$\check{\mathcal{N}}(v) = \overline{\mathcal{N}(v) \smallsetminus \bigcup_{\gamma \subset \{L=0\}} N(\gamma^*)}$$

and

$$F_v = F_t \cap \check{\mathcal{N}}(v).$$

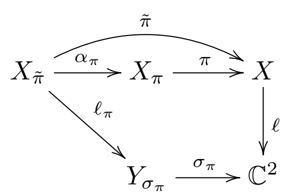
The intersection $T_{\epsilon,\eta} \cap \check{\mathcal{N}}(v)$ is fibred by the oriented circles obtained by intersecting the disc-fibers of $N(E_v)$ with $T_{\epsilon,\eta}$. Then the monodromy $\phi\colon F_t \to F_t$ is defined on each $F_v$ as the diffeomorphism of first return of these circles, so it is a periodic diffeomorphism.

Now consider an edge $e = [v, v']$ in $\Gamma_\pi$ and let $p \in E_v \cap E_{v'}$ be the corresponding intersection point. Let $\mathcal{N}(e)$ be the component of $N(E_v) \cap N(E_{v'})$ containing $p$. The intersection $F_e = F_t \cap \mathcal{N}(e)$ is the disjoint union of $\gcd(m_v, m_{v'})$ annuli which are cyclically exchanged by $\phi$. Its monodromy is related to the length of the edge $e$ by the following proposition.

**Proposition 4.12.** *(*[MMA11, Theorem 7.3, (iv)]*) Denote by $\phi_e$ the restriction $\phi|_{F_e}\colon F_e \to A_e$ of the monodromy $\phi$ to $F_e$. Then $\phi_e^{m_v m_{v'}} = id_{F_e}$ and $\phi_e$ has screw number $-\frac{1}{m_v m_{v'}}$. In other words, this screw number is the opposite of* $\mathrm{length}(e)$.

4.5. **The resolution adapted to a projection.** Let $\ell\colon (X,0) \to (\mathbb{C}^2,0)$ be a generic projection as in Section 2.2 and let $\pi\colon X_\pi \to X$ be a good resolution of $(X,0)$. Consider the minimal sequence $\sigma_\pi\colon Y_{\sigma_\pi} \to \mathbb{C}^2$ of blowups of $(\mathbb{C}^2,0)$ such that $\widetilde{\ell}\big(V(\Gamma_\pi)\big) \subset V(\Gamma_\sigma)$.

Now let $\tilde{\pi}\colon X_{\tilde{\pi}} \to X$ be the minimal good resolution of $(X,0)$ obtained by pulling back $\sigma_\pi$, through $\ell$, normalizing the resulting surface, and then resolving the remaining singularities. Denote by $\ell_\pi\colon X_{\tilde{\pi}} \to Y_{\sigma_\pi}$ the projection morphism, so that $\sigma_\pi \circ \ell_\pi = \ell \circ \tilde{\pi}$. The resolution $\tilde{\pi}$ factors through $\pi$, and if we denote by $\alpha_\pi\colon X_{\tilde{\pi}} \to X_\pi$ the resulting morphism we obtain the following commutative diagram:




By construction, the following properties are satisfied:

(i) For every vertex $v$ of $\Gamma_\pi$, we have $\tilde{\ell}(v) \in V(\Gamma_{\sigma_\pi})$;
(ii) For every vertex $\nu$ of $\Gamma_{\sigma_\pi}$, we have $\tilde{\ell}^{-1}(\nu) \subset V(\Gamma_{\tilde{\pi}})$.

**Definition 4.13.** We call the map $\tilde{\pi}\colon X_{\tilde{\pi}} \to X$ defined as above the resolution of $(X,0)$ *adapted to $\pi$ and $\ell$.*

As part of the data of the adapted resolution we also keep the morphisms $\sigma_\pi\colon Y_{\sigma_\pi} \to \mathbb{C}^2$ and $\ell_\pi\colon X_{\tilde{\pi}} \to Y_{\sigma_\pi}$.

4.6. **The local degree formula.** Let $(X,0)$ be a complex surface germ. For each divisorial point $v \in NL(X,0)$, we are going to define the local degree $\deg(v) \in \mathbb{N}^*$ of a generic projection $\ell\colon (X,0) \to (\mathbb{C}^2,0)$ at $v$.

Fix a divisorial point $v$ of $\mathrm{NL}(X,0)$ and let $\pi\colon X_\pi \to X$ be a good resolution of $(X,0)$ such that $v$ is associated with a component $E_v$ of the exceptional divisor $\pi^{-1}(0)$. In the notation of Section 4.5, let $\tilde{\pi}\colon X_{\tilde{\pi}} \to X$ be the resolution adapted to $\pi$ and $\ell$ and let $\ell_\pi\colon X_{\tilde{\pi}} \to Y_{\sigma_\pi}$ be the natural morphism. Set $\nu = \widetilde{\ell}(v)$ and let us denote by $C_\nu$ the component of $\sigma_\pi^{-1}(0)$ associated with $\nu$, so that we have $\ell_\pi(E_v) = C_\nu$. For each component $E_i$ of $\tilde{\pi}^{-1}(0)$ (respectively $C_j$ of $\sigma_\pi^{-1}(0)$), let us choose a normal disc-bundle $N(E_i)$ (resp. $N(C_j)$). We use again the notations

$\mathcal{N}(v)$ and $\mathcal{N}(\nu)$ introduced in Section 4.4. We can then adjust the disc bundles $N(E_i)$ and $N(C_j)$ in such a way that the cover $\ell$ restricts to a cover

$$\ell_v \colon \tilde{\pi}\big(\mathcal{N}(v)\big) \to \sigma_\pi\big(\mathcal{N}(\nu)\big)$$

branched precisely on the polar curve of $\ell$ (if $v$ is not a $\mathcal{P}$-node, the branching locus is just the origin). This is what allows us to define the local degree of $v$.

**Definition 4.14.** The *local degree* $\deg(v)$ of $v$ is defined as the degree $\deg(v) = \deg(\ell_v)$ of the cover $\ell_v$. This integer does not depend on the choice of a generic projection $\ell$.

We will now prove two simple lemmas about the local degrees.

**Lemma 4.15.** *Let $\pi \colon X_\pi \to X$ be a good resolution of $(X,0)$ that factors through the blowup of the maximal ideal and through the Nash transform of $(X,0)$ and let $N$ be the set of nodes of $\Gamma_\pi$ (that is, the subset of $V(\Gamma_\pi)$ consisting of the $\mathcal{P}$-nodes, the $\mathcal{L}$-nodes, and of all the vertices of genus strictly grater than zero or whose valence in $\Gamma_\pi$ is at least three). Then* $\deg$ *is constant on each connected component of $\Gamma_\pi \smallsetminus N$.*

*Proof.* Given a string $\mathcal{S}$ of $\Gamma_\pi$, we set $\mathcal{N}(\mathcal{S}) = N(e)$ if $\mathcal{S}$ is an edge $e$ of $\Gamma_\pi$, and $\mathcal{N}(\mathcal{S}) = \bigcup_v N(E_v)$, where the union is taken over the set of vertices of $\Gamma_\pi$ contained in $\mathcal{S}$, otherwise. By definition, a connected component $\mathcal{S}$ of $\Gamma_\pi \smallsetminus N$ is a string of $\Gamma_\pi$. In particular, as $\mathcal{S}$ contains no $\mathcal{P}$-nodes, the polar curve of $\ell$ does not intersect $\mathcal{N}(\mathcal{S}) \smallsetminus \{0\}$, and so the restriction $\ell_\mathcal{S} = \ell|_{\mathcal{N}(\mathcal{S})}$ of $\ell_\mathcal{S}$ to $\mathcal{N}(\mathcal{S})$ is a regular cover outside 0. Denote by $d$ the degree of this cover. If $v$ is any divisorial point in the interior of $\mathcal{S}$, by further blowing up double points of the exceptional divisor we can assume without loss of generality that $v$ is a vertex of $\Gamma_\pi$. Then $\ell_v$ is the restriction of $\ell_\mathcal{S}$ to $\mathcal{N}(E_v)$, and therefore $\deg(v) = d$. □

**Remark 4.16.** Although this will not be needed in the paper, it is worth noticing that Lemma 4.15 allows us to extend the local degree to a map $\deg \colon \mathrm{NL}(X,0) \to \mathbb{Z}$ on the whole non-archimedean link $\mathrm{NL}(X,0)$. This map can also be characterized as the unique upper semi-continuous map on $\mathrm{NL}(X,0)$ that takes the value $\deg(v)$ on any divisorial valuation $v$.

As a consequence of Lemma 4.15, if $\pi$ is a resolution of $(X,0)$ that factors through the blowup of the maximal ideal and through the Nash transform then we can define $\deg(e)$ for any edge $e$ of the dual graph $\Gamma_\pi$ by setting $\deg(e) = \deg(v)$, where $v$ is any divisorial point in the interior of $e$. The following result gives an alternative way to compute local degrees.

**Lemma 4.17.** *Let $\pi \colon X_\pi \to X$ be a good resolution of $(X,0)$, let $\ell \colon (X,0) \to (\mathbb{C}^2,0)$ be a generic projection, let $\tilde{\pi} \colon X_{\tilde{\pi}} \to X$ be the resolution of $(X,0)$ adapted to $\pi$ and $\ell$, let $v$ be a vertex of $\Gamma_\pi$, and let $e$ be an edge in $\Gamma_\pi$ adjacent to $v$. Denote by $W_e$ the set of the edges $e'$ of $\Gamma_{\tilde{\pi}}$ that are adjacent to $v$ and such that $\widetilde{\ell}(e') = \widetilde{\ell}(e)$. Then we have*

$$\deg(v) = \sum_{e' \in W_e} \deg(e').$$

*Proof.* Write $e = [v,w]$ and set $\tilde{v} = \tilde{\ell}(v)$ and $\tilde{w} = \tilde{\ell}(w)$. Then $\tilde{\ell}(e)$ is a string joining $\tilde{v}$ to $\tilde{w}$ in $\Gamma_{\sigma_\pi}$, and the intersection $F'_{\tilde{\ell}(e)} = F'_t \cap \sigma_\pi\big(\mathcal{N}(\tilde{\ell}(e))\big)$ is a disjoint union of annuli. For each edge $e' = [v,w']$ of $W_e$, the intersection $F_{e'} = F_t \cap \pi\big(\mathcal{N}(e')\big)$ is a disjoint union of annuli. After adjusting the bundles $N(E_i)$ and $N(C_j)$ if necessary, we can assume that the cover $\ell$ restricts to a (possibly disconnected) regular cover

$$\ell' \colon \bigcup_{e' \in W_e} F_{e'} \to F'_{\tilde{\ell}(e)}.$$

By computing the degree of $\ell'$ over a point on the boundary of $F'_{\tilde{\ell}(e)}$ that is inside $N_{C_{\tilde{v}}}$, we see that $\deg(\ell') = \deg(v)$. On the other hand, computing the degree of $\ell'$ over a point on the boundary of $F'_{\tilde{\ell}(e)}$ that is inside $N_{C_{\tilde{w}}}$, we obtain

$$\deg(\ell') = \sum_{e' \in W_e} \deg(\ell|_{F_{e'}}) = \sum_{e' \in W_e} \deg(w').$$

Finally, by blowing-up all the intersections $E_v \cap E_{v'}$ where $v, v'$ belong to the set of nodes $N$ of $\Gamma_\pi$, we can assume that there are no adjacent nodes in $\Gamma_\pi$. Therefore, for each edge $e' = [v, w']$ of $W_e$, the vertex $w'$ has valency 2 vertex, $\mathcal{N}(w')$ is a disjoint union of annuli each of whom has common boundary with one of the annuli of $F_{e'}$, and so we have $\deg(e') = \deg(w')$. This proves the Lemma. □

The following proposition, which we call the local degree formula, will play a key role in our proof of Theorem 4.2.

**Proposition 4.18.** *Let $\pi \colon X_\pi \to X$ be a good resolution of $(X, 0)$ which factors through the blowup of the maximal ideal and through the Nash transform of $(X, 0)$ and let $e$ be an edge of $\Gamma_\pi$. Then*

$$\mathrm{length}\big(\widetilde{\ell}(e)\big) = \deg(e)\mathrm{length}(e).$$

Observe that $\widetilde{\ell}(e)$ is in general not an edge of a resolution of $(\mathbb{C}^2, 0)$, and here we consider its length for the skeletal metric of $\mathrm{NL}(\mathbb{C}^2, 0)$.

*Proof.* Denote by $v$ and $v'$ the two vertices of $\Gamma_\pi$ adjacent to $e$. Let $z_1$ and $z_2$ be two generic linear forms for $(X, 0)$ such that the projection $\ell = (z_1, z_2) \colon (X, 0) \to (\mathbb{C}^2, 0)$ is generic, let $\tilde{\pi} \colon X_{\tilde{\pi}} \to X$ be the resolution of $(X, 0)$ adapted to $\pi$ and $\ell$, and consider the commutative diagram $\ell \circ \tilde{\pi} = \ell_\pi \circ \sigma_\pi$ introduced in Section 4.5. Consider the string $\widetilde{\ell}(e)$ of the dual tree $\Gamma_{\sigma_\pi}$ of $\sigma_\pi$, and name its vertices $\nu_1, \ldots, \nu_n$ following the order of the string from $\nu_1 = \widetilde{\ell}(v)$ to $\nu_n = \widetilde{\ell}(v.)$.

We can take a Milnor ball for $(X, 0)$ of the form $B_\epsilon = \big\{z \in \mathbb{C}^n \,\big|\, |z_1| \leq \epsilon, \|z\| \leq R\epsilon\big\}$, for a suitable $R > 0$ (see [BNP14, Section 4]), and consider the ball $B'_\epsilon = \ell(B_\epsilon)$ in $\mathbb{C}^2$.

Write again $\sigma_\pi\big(\mathcal{N}(\mathcal{S})\big) = B'_\epsilon \cap \sigma_\pi\big(\mathcal{N}(\mathcal{S})\big)$ and $\pi\big(\mathcal{N}(e)\big) = B_\epsilon \cap \pi\big(\mathcal{N}(e)\big)$, so that $\deg(e)$ is by definition the degree of the restriction $\ell|_{\pi(\mathcal{N}(e))} \colon \pi\big(\mathcal{N}(e)\big) \to \sigma_\pi\big(\mathcal{N}(\mathcal{S})\big)$.

Let us choose a linear form $h \colon (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ such that $L = h \circ \ell$ is a generic linear form for $(X, 0)$. Let $t \in D^*_\eta$ and let $F_t = L^{-1}(t) \cap B_\epsilon$ be the Milnor–Lê fiber of $L \colon (X, 0) \to (\mathbb{C}, 0)$ and let $F'_t = h^{-1}(t) \cap B'_\epsilon$ be that of $h \colon (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$. Then we have $\ell(F_t) = F'_t$.

Set $F_e = F_t \cap \pi\big(\mathcal{N}(e)\big)$ and $F_{\mathcal{S}} = F'_t \cap \sigma_\pi\big(\mathcal{N}(\mathcal{S})\big)$. The cover $\ell$ restricts to a regular cover $\ell_e \colon F_e \to F_{\mathcal{S}}$.

On the one hand, $F_e$ is a disjoint union of $K = \gcd(m_v, m_{v'})$ annuli, and by Proposition 4.12, the monodromy $\phi \colon F_t \cap B_\epsilon \to F_t \cap B_\epsilon$ of the Milnor–Lê fiber of

$L\colon (X,0) \to (\mathbb{C},0)$ restricts to a map $\phi_e\colon F_e \to F_e$ with screw number

$$\tau = -\frac{1}{m_v m_{v'}} = -\mathrm{length}(e).$$

On the other hand, for each $i = 1, \ldots, n-1$, denote by $e_i$ the edge $e_i = (\nu_i)-(\nu_{i+1})$. The number $K' = \gcd(m_{\nu_i}, m_{\nu_{i+1}})$ does not depend on the choice of $i$, as can be verified by computing the intersection number on $Y_{\sigma_\pi}$ of the divisor $\sum_{j=1}^{n} m_{\nu_j} E_{\nu_j}$ with each of the $E_{\nu_i}$, for $i = 2, \ldots, n-1$. By Proposition 4.12, $F_{e_i} = F'_t \cap \sigma_\pi\big(\mathcal{N}(e_i)\big)$ is is a disjoint union of $K'$ annuli which are cyclically exchanged by the monodromy $\phi'\colon F'_t \cap B'_\epsilon \to F'_t \cap B'_\epsilon$ of the Milnor–Lê fiber of $h$ and the restriction of $\phi'|_{F_{e_i}}\colon F_{e_i} \to F_{e_i}$ has screw number $-\mathrm{length}(e_i)$.

Now, $F_{\mathcal{S}}$ is a disjoint union of $K'$ annuli and the intersections $F_{e_i} = F'_t \cap \sigma_\pi\big(\mathcal{N}(e_i)\big)$, for $i = 1, \ldots, n-1$, are concentric unions of $K'$ annuli inside $F_{\mathcal{S}}$. Therefore, applying Lemma 4.10 we obtain that the restriction $\phi'|_{F_{\mathcal{S}}}\colon F_{\mathcal{S}} \to F_{\mathcal{S}}$ has screw number

$$\tau' = -\sum_{i=1}^{n-1} \mathrm{length}(e_i) = -\mathrm{length}(\mathcal{S}).$$

A representative of the monodromy $\phi'\colon F'_t \to F'_t$ is the first return diffeomorphism on $F'_t$ of a 1-dimensional flow transversal to all the Milnor fibers, lifted from the standard unit tangent vector field to the circle $S^1_{|t|}$. Since $L = h \circ \ell$ then a representative of the monodromy $\phi\colon F_t \to F_t$ of the Milnor–Lê fibration of $L$ is obtained by lifting this flow by $\ell$ and by taking its first return diffeomorphism on $F_t$. Therefore, we have $\ell \circ \phi = \phi' \circ \ell$. Moreover, the restriction $\ell|_{F_e} F_e \to F_{\mathcal{S}}$ is a cover with degree $\deg(e)$. Applying Lemma 4.11 to ${\phi'}^{K'}$ we then obtain $\tau' = \deg(e)\tau$. This completes the proof. □

4.7. **Lifting $\Delta$ under $\widetilde{\ell}$.** We will now explain how to use adapted resolutions and the local degree formula to relate the Laplacian of $\mathcal{I}_X$ to that of $\mathcal{I}_{\mathbb{C}^2}$ via the generic projection $\widetilde{\ell}$.

**Proposition 4.19.** *Let $\ell\colon (X,0) \to (\mathbb{C}^2,0)$ be a generic projection, let $\pi$ be a good resolution of $(X,0)$ which factors through the blowup of the maximal ideal and through the Nash transform of $(X,0)$, and let $\tilde{\pi}\colon X_{\tilde{\pi}} \to X$ be the good resolution of $(X,0)$ adapted to $\pi$ and $\ell$. For each vertex $v$ of $V(\Gamma_\pi)$, we have*

$$\Delta_{\Gamma_{\tilde{\pi}}}\big(\mathcal{I}_X\big)(v) = \deg(v)\Delta_{\Gamma_{\sigma_\pi}}\big(\mathcal{I}_{\mathbb{C}^2}\big)\big(\widetilde{\ell}(v)\big).$$

*Proof.* Let $\mathrm{St}_v$ denote the set of those vertices of $\Gamma_{\tilde{\pi}}$ that are adjacent to $v$. If $v' \in \mathrm{St}_v$, we denote by $e_{v,v'}$ the edge between $v$ and $v'$. We have:

$$\Delta_{\Gamma_{\tilde{\pi}}}\big(\mathcal{I}_X\big)(v) = \sum_{v' \in St_v} \frac{\mathcal{I}_X(v') - \mathcal{I}_X(v)}{\mathrm{length}(e_{v,v'})}.$$

By definition of the adapted resolution, $\widetilde{\ell}(St_v)$ is a subset of the set of vertices of $\Gamma_{\sigma_\pi}$; let us denote by $\nu_1, \ldots, \nu_r$ its elements. For each $i = 1, \ldots, r$, the simple path in $\Gamma_{\sigma_\pi}$ joining $\widetilde{\ell}(v)$ to $\nu_i$ is a string $\mathcal{S}_i$. If $v' \in \mathrm{St}_v$ is such that $\nu_i = \widetilde{\ell}(v')$, then $\mathcal{S}_i = \widetilde{\ell}(e_{v,v'})$, and by Proposition 4.18 we have

$$\deg(e_{v,v'})\mathrm{length}(e_{v,v'}) = \mathrm{length}(\mathcal{S}_i).$$

By Lemma 3.9 we have $\mathcal{I}_X(v) = \mathcal{I}_{\mathbb{C}^2}\big(\widetilde{\ell}(v)\big)$ and $\mathcal{I}_X(v') = \mathcal{I}_{\mathbb{C}^2}(\nu_i)$ for all $v' \in \mathrm{St}_v$ such that $\nu_i = \widetilde{\ell}(v')$. Then, writing $\mathrm{St}_v$ as the disjoint union of the $\widetilde{\ell}^{-1}(\nu_i)$, for $i = 1, \ldots, r$, we obtain:

$$\Delta_{\Gamma_{\tilde{\pi}}}\big(\mathcal{I}_X\big)(v) = \sum_{i=1}^{r} \Big( \sum\nolimits_{v' \in \widetilde{\ell}^{-1}(\nu_i)} \deg(e_{v,v'}) \Big) \frac{\mathcal{I}_{\mathbb{C}^2}(\nu_i) - \mathcal{I}_{\mathbb{C}^2}\big(\widetilde{\ell}(v)\big)}{\mathrm{length}(\mathcal{S}_i)}.$$

By Lemma 4.17, for every $i \in 1, \ldots, r$ we have $\sum_{v' \in \widetilde{\ell}^{-1}(\nu_i)} \deg(e_{v,v'}) = \deg(v)$. Since $\mathcal{I}_{\mathbb{C}^2}$ is linear on the strings $\mathcal{S}_i$ by Corollary 4.8, we deduce that

$$\Delta_{\Gamma_{\tilde{\pi}}}\big(\mathcal{I}_X\big)(v) = \deg(v) \sum_{i=1}^{r} \frac{\mathcal{I}_{\mathbb{C}^2}(\nu_i) - \mathcal{I}_{\mathbb{C}^2}\big(\widetilde{\ell}(v)\big)}{\mathrm{length}(\mathcal{S}_i)} = \deg(v) \Delta_{\Gamma_{\sigma_\pi}}\big(\mathcal{I}_{\mathbb{C}^2}\big)\big(\widetilde{\ell}(v)\big),$$

which proves the proposition. □

4.8. **Lifting $K$ under $\widetilde{\ell}$.** The last ingredient we need for the proof of Theorem 4.2 is a tool to compute also the canonical divisor of a dual resolution graph of our singular surface via a generic projection.

**Proposition 4.20.** *Let $\ell \colon (X,0) \to (\mathbb{C}^2,0)$ be a generic projection, let $\pi$ be a good resolution of $(X,0)$ which factors through the blowup of the maximal ideal and through the Nash transform of $(X,0)$, let $\tilde{\pi} \colon X_{\tilde{\pi}} \to X$ be the good resolution of $(X,0)$ adapted to $\pi$ and $\ell$, and let $v$ be a vertex of $\Gamma_\pi$. Then*

$$K_{\Gamma_{\tilde{\pi}}}(v) - m_v p_v = \deg(v) K_{\Gamma_{\sigma_\pi}}\big(\widetilde{\ell}(v)\big).$$

*Proof.* We use again the notations of Sections 4.4 and 4.6: we write $E_v$ for the divisor on $X_{\tilde{\pi}}$ associated with $v$, and $C_\nu$ for the divisor on $Y_{\sigma_\pi}$ associated with $\nu = \widetilde{\ell}(v)$, and again we consider the cover $\ell_v \colon \mathcal{N}(E_v) \to \mathcal{N}(C_\nu)$ introduced in Section 4.6. We choose again a linear form $h \colon (\mathbb{C}^2,0) \to (\mathbb{C},0)$ on $(\mathbb{C}^2,0)$ such that $L = \ell \circ h \colon (X,0) \to (\mathbb{C},0)$ is a generic linear form on $(X,0)$, as was done in the proof of the degree formula (Proposition 4.18). For $t > 0$ small enough, let $F_t = L^{-1}(t) \cap B_\epsilon$ be the Milnor–Lê fiber of $L$ and let $F'_t = h^{-1}(t) \cap B'_\epsilon$ be that of $h$, so that we have $\ell(F_t) = F'_t$. Set $F_v = F_t \cap \mathcal{N}(E_v)$ and $F'_\nu = F'_t \cap \mathcal{N}(E_\nu)$. Then $\ell_v$ restricts to a map $\ell_v|_{F_v} \colon F_v \to F'_\nu$.

Let $l_v$ (respectively $p_v$) be the number of irreducible components of $L = 0$ (resp. of the polar curve of $\ell$) whose strict transforms by $\pi$ intersects $E_v$. In particular, $l_v > 0$ (resp. $p_v > 0$) if and only if $v$ is an $\mathcal{L}$-node (resp. a $\mathcal{P}$-node).

The cover $\ell_v \colon F_v \to F'_\nu$ is branched on $p_v m_v$ points of $F_v$. Since $\ell$ is generic, then the images of these points are $p_v m_v$ distinct points of $F'_\nu$ and if $p \in F'_\nu$ is one of them, $\ell_v^{-1}(p)$ consists of $\deg(v) - 1$ points. Therefore, applying Hurwitz formula we obtain $\chi(F_v) - \big(\deg(v) - 1\big) m_v p_v = \deg(v) \big(\chi(F'_\nu) - m_v p_v\big)$, that is

$$\chi(F_v) + m_v p_v = \deg(v) \chi(F'_\nu). \tag{2}$$

Let us identify $F_v$ with its pull-back by $\tilde{\pi}$ and let us consider again the generic linear form $L \colon (X,0) \to (\mathbb{C}^2,0)$. If $p$ is a smooth point of $E_v$ which does not intersect the strict transform $L^*$ of $L$ on $X_{\tilde{\pi}}$, the total transform of $L$ is defined by an equation of the form $h(u,v) = u^{m_v}$ in suitable local coordinates $(u,v)$ centered at $p$. Therefore, the map $\mathcal{N}(E_v) \to E_v \cap \mathcal{N}(E_v)$ which sends each fiber-disc of $\mathcal{N}(E_v)$ to its intersecting point with $E_v$ restricts to a regular cover $\rho_v \colon F_v \to E_v \cap \mathcal{N}(E_v)$ of degree $m_v$. Then, applying Hurwitz formula again, we obtain $\chi(F_v) = m_v \chi\big(E_v \cap \mathcal{N}(E_v)\big)$. By the same argument, we also have $\chi(F'_\nu) = m_\nu \chi\big(C_\nu \cap \mathcal{N}(C_\nu)\big)$.

Assume first that $v$ is not an $\mathcal{L}$-node, that is $l_v = 0$. Then $\chi\big(E_v \cap \mathcal{N}(E_v)\big) = 2 - 2g(v) - \mathrm{val}(v)$. This implies that $\chi(F_v) = -K_{\Gamma_\pi}(v)$. By the same argument, we have $\chi(F'_\nu) = -K_{\Gamma_{\sigma_\pi}}(\nu)$. Using equation (2), we deduce that $K_{\Gamma_\pi}(v) - m_v p_v = \deg(v) K_{\Gamma_{\sigma_\pi}}(\nu)$, which is what we wanted.

Assume now that $v$ is an $\mathcal{L}$-node. Then $\chi\big(E_v \cap \mathcal{N}(E_v)\big) = 2 - 2g(v) - \mathrm{val}(v) - l_v$, which implies $\chi(F_v) = -K_{\Gamma_\pi}(v) - m_v l_v$, and since $\nu$ is the root vertex of $\Gamma_{\sigma_\pi}$, we have $m_\nu = 1$, and we get $\chi(F'_\nu) = \chi(C_\nu) - 1 = -K_{\Gamma_{\sigma_\pi}}(\nu) - 1$. Using equation (2), we then obtain $K_{\Gamma_\pi}(v) - m_v p_v + m_v l_v = \deg(v)\big(K_{\Gamma_{\sigma_\pi}}(\nu) + 1\big)$, that is

$$K_{\Gamma_\pi}(v) - m_v p_v = \deg(v) K_{\Gamma_{\sigma_\pi}}(\nu) + \big(\deg(v) - m_v l_v\big). \tag{3}$$

Since $v$ is an $\mathcal{L}$-node, then $\nu = \widetilde{\ell}(v)$ is the root vertex of the tree $\Gamma_{\sigma_\pi}$. Let $(\gamma, 0) \subset (\mathbb{C}^2, 0)$ be a generic complex line. Its strict transform $\gamma^*$ by $\sigma_\pi$ is a curvette of $E_\nu$ and since $m_\nu = 1$, the intersection $\gamma \cap F'_v$ consists of a single point $p$. The cardinal of $\ell_v^{-1}(p)$ is equal to the degree $\deg(v)$ of the cover $\ell_v$ and also to the number of points in the intersection $\ell^{-1}(\gamma) \cap F_v$, which is exactly $m_v l_v$. This proves that $\deg(v) - m_v l_v = 0$. Replacing this in equation (3), we obtain $K_{\Gamma_\pi}(v) - m_v p_v = \deg(v) K_{\Gamma_{\sigma_\pi}}(\nu)$, which is what we wanted to prove. □

4.9. **Proof of Theorem 4.2.** We are finally ready to put together all the pieces we prepared so far and finish the proof of our main theorem.

Thanks to Lemma 4.6, it suffices to prove the theorem for a good resolution $\pi$ of $(X, 0)$ that factors through the blowup of the maximal ideal and through the Nash transform of $(X, 0)$. Again by the same lemma, it's enough to prove if for the resolution $\tilde{\pi}$ of $(X, 0)$ adapted to $\pi$ and to a given generic linear form $\ell$ on $(X, 0)$. As the formula can be easily verified at all vertices of $\Gamma_{\tilde{\pi}}$ that are not vertices of $\Gamma_\pi$ thanks to Lemma 3.8 by an argument identical to the one in the proof of Lemma 4.6, we just have to prove that the formula holds for $\Delta_{\Gamma_{\tilde{\pi}}}(\mathcal{I}_X|_{\Gamma_{\tilde{\pi}}})$ at a vertex $v$ of $\Gamma_\pi$. We want to prove that

$$\Delta_{\Gamma_{\tilde{\pi}}}(\mathcal{I}_X)(v) - K_{\Gamma_{\tilde{\pi}}}(v) - 2 m_v l_v + m_v p_v = 0.$$

Set $\nu = \widetilde{\ell}(v)$. As we have already proved the theorem for the modification $\sigma_\pi$ of $(\mathbb{C}^2, 0)$ in Proposition 4.7, we have $\Delta_{\Gamma_{\sigma_\pi}}(\mathcal{I}_{\mathbb{C}^2})(\nu) - K_{\Gamma_{\sigma_\pi}}(\nu) - 2 = 0$ if $\nu = \mathrm{ord}_0$ is the root of $\mathrm{NL}(\mathbb{C}^2, 0)$, while $\Delta_{\Gamma_{\sigma_\pi}}(\mathcal{I}_{\mathbb{C}^2})(\nu) - K_{\Gamma_{\sigma_\pi}}(\nu) = 0$ otherwise. By Proposition 4.19, we have $\Delta_{\Gamma_{\tilde{\pi}}}(\mathcal{I}_X)(v) = \deg(v) \Delta_{\Gamma_{\sigma_\pi}}(\mathcal{I}_{\mathbb{C}^2})(\nu)$, while by Proposition 4.20 we have $K_{\Gamma_{\tilde{\pi}}}(v) - m_v p_v = \deg(v) K_{\Gamma_{\sigma_\pi}}(\nu)$. Therefore, we have

$$\Delta_{\Gamma_{\tilde{\pi}}}(\mathcal{I}_X)(v) - K_{\Gamma_{\tilde{\pi}}}(v) - 2 m_v l_v + m_v p_v = \deg(v)\big(\Delta_{\Gamma_{\sigma_\pi}}(\mathcal{I}_{\mathbb{C}^2})(\nu) - K_{\Gamma_{\sigma_\pi}}(\nu)\big) - 2 m_v l_v.$$

If $v$ is not an $\mathcal{L}$-node, then $l_v = 0$ and we get

$$\Delta_{\Gamma_{\tilde{\pi}}}(\mathcal{I}_X)(v) - K_{\Gamma_{\tilde{\pi}}}(v) - 2 m_v l_v + m_v p_v = \deg(v)\big(\Delta_{\Gamma_{\sigma_\pi}}(\mathcal{I}_{\mathbb{C}^2})(\nu) - K_{\Gamma_{\sigma_\pi}}(\nu)\big) = 0.$$

If $v$ is an $\mathcal{L}$-node, then $\nu = \mathrm{ord}_0$ and we obtain:

$$\Delta_{\Gamma_{\tilde{\pi}}}(\mathcal{I}_X)(v) - K_{\Gamma_{\tilde{\pi}}}(v) - 2 m_v l_v + m_v p_v = \deg(v)\big(\Delta_{\Gamma_{\sigma_\pi}}(\mathcal{I}_{\mathbb{C}^2})(\nu) - K_{\Gamma_{\sigma_\pi}}(\nu) - 2\big) = 0.$$

This completes the proof of Theorem 4.2. □

## 5. Application: from global geometric data to inner rates

The main result of this section, which is a consequence of Theorem 4.2, explains in which sense the metric structure of the germ $(X, 0)$, which is a very local datum, is determined by more global geometric data, namely by the topology of a good resolution of $(X,0)$ and the position of the components of a generic hyperplane section of $(X,0)$ and of the components of the polar curve of a generic projection of $(X,0)$ onto $(\mathbb{C}^2,0)$.

**Corollary 5.1.** *Let $(X,0)\subset(\mathbb{C}^n,0)$ be an isolated complex surface singularity and let $\pi\colon X_\pi\to X$ be the minimal good resolution of $(X,0)$ that factors through the blowup of the maximal ideal and through the Nash transform of $(X,0)$. Then the inner rate function $\mathcal{I}_X$ on the whole of $\mathrm{NL}(X,0)$, and hence the local inner metric structure of the germ $(X,0)$, is determined by the following data:*

- *(i) the topological data consisting of the dual graph $\Gamma_\pi$ decorated with the Euler classes and the genera of its vertices;*
- *(ii) on each vertex $v$ of $\Gamma_\pi$, an arrow for each irreducible component of a generic hyperplane section of $(X,0)$ whose strict transform on $X_\pi$ passes through the prime divisor $E_v$;*
- *(iii) on each vertex $v$ of $\Gamma_\pi$, an arrow for each irreducible component of a generic generic polar curve of $(X,0)$ whose strict transform on $X_\pi$ passes through the prime divisor $E_v$.*

Observe that we do not need to require the metric on $\Gamma_\pi$ to be part of the initial data. Indeed, the multiplicities $m_v$ which we use to define it can be deduced from the data given in (i) and (ii), as will be explained in the proof.

*Proof.* First observe that the graph $\Gamma_\pi$ decorated by the two sets of arrows described in the statement determines the numbers $l_v$, $p_v$, $g(v)$, and $\mathrm{val}_{\Gamma_\pi}(v)$. Number the vertices of $\Gamma_\pi$ as $v_1,\ldots,v_n$, and denote by $M=(E_{v_i}\cdot E_{v_j})_{1\le i,j\le n}$ the intersection matrix of the exceptional divisor $\pi^{-1}(0)$ of $\pi$. Let $h\colon(X,0)\to(\mathbb{C}^2)$ be a generic linear form on $(X,0)$ and let $h^*$ be the strict transform of $h$ by $\pi$. Since the total transform $(h)$ of $h$ by $\pi$, which is $(h)=\sum_{i=1}^n m_{v_i}E_{v_i}+h^*$, is a principal divisor (see [Lau71, Theorem 2.6]), then for every vertex $v$ of $\Gamma_\pi$ we have $(h).E_v=0$, that is

$$\sum_{[v,v']} m_{v'}+m_vE_v^2+l_v=0, \tag{4}$$

where the sum runs over the edges of $\Gamma_\pi$ adjacent to $v$ and as usual $l_v$ denotes the number of the arrows at $v$ considered in part (ii) of the statements. By combining the equations (4) for $v=v_1,\ldots,v_n$ we obtain the linear system $M\underline{m}=-\Lambda$, where $\underline{m}={}^t(m_{v_1},\ldots,m_{v_n})$ is the vector of the multiplicities of the exceptional divisors and $\Lambda={}^t(l_{v_1},\ldots,l_{v_n})$. The matrix $M$, being negative definite, is invertible, and therefore we can retrieve the vector of multiplicities as $\underline{m}=-M^{-1}\Lambda$. This implies that the metric on $\Gamma_\pi$, as well as the divisors $L$, $P$, and $K_\pi$ on $\mathrm{NL}(X,0)$, are determined by the data described in the statement. By Theorem 4.2 we therefore know the Laplacian $\Delta_{\Gamma_\pi}(\mathcal{I}_X|_{\Gamma_\pi})$ of the inner rate function on $\Gamma_\pi$. We will now explain how to deduce the inner rate $q_v$ for every vertex $v$ of $\Gamma_\pi$. Consider the coefficient of the Laplacian at a vertex $v$, as given by the formula of Theorem 4.2, and multiply both sides by $m_v$. This yields the equality

$$\sum_{[v,v']} m_{v'}(q_{v'}-q_v)=\mathrm{val}_{\Gamma_\pi}(v)+2g(E_v)-2+2l_v-p_v,$$

which combined with equation (4) gives

$$q_vm_vE_v^2+\sum_{[v,v']} m_{v'}q_{v'}=\mathrm{val}_{\Gamma_\pi}(v)+2g(E_v)-2+(2-q_v)l_v-p_v.$$

Now, observe that $l_v$ vanishes unless $v$ is an $\mathcal{L}$-node of $(X,0)$, in which case $q_v = 1$, so that for every $v$ we have $(2-q_v)l_v = l_v$. We have obtained the linear system $M\underline{a} = \underline{b}$, where $\underline{a} = {}^t(q_{v_1}m_{v_1}, \ldots, q_{v_n}m_{v_n})$ and $\underline{b} = {}^t(\mathrm{val}_{\Gamma_\pi}(v_1) + 2g(E_{v_1}) - 2 + l_{v_1} - p_{v_1}, \ldots, \mathrm{val}_{\Gamma_\pi}(v_n) + 2g(E_{v_n}) - 2 + l_{v_n} - p_{v_n})$. We then have $\underline{a} = M^{-1}\underline{b}$, and therefore we obtain the inner rates $q_v$ by dividing each entry of $\underline{a}$ by the corresponding multiplicity $m_v$. By linearity on the edges this determines the inner rate function $\mathcal{I}_X$ on the dual graph $\Gamma_\pi$. It remains to show that $\mathcal{I}_X$ is completely determined by its restriction to $\Gamma_\pi$. This is the content of the next proposition. □

The next result will require us to consider a different metric $d$ on $\mathrm{NL}(X,0)$. This is defined on a dual graph $\Gamma_\pi$ in a similar way as the one introduced in Section 2.1, by declaring the length of an edge connecting two divisorial valuations $v$ and $v'$ to be equal to $\mathrm{lcm}\{m_v, m_{v'}\}^{-1} = \gcd\{m_v, m_{v'}\}/m_v m_{v'}$. Being stable under further blowup by a computation analogous to the one performed in Section 2.1, these metrics on dual graphs induce a metric $d$ on $\mathrm{NL}(X,0)$. The difference between the two metrics has also been discussed in [Jon15, Section 7.4.10] and, in a more arithmetic setting, in [BN16, Remark 2.3.4]. Recall that $r_\pi \colon \mathrm{NL}(X,0) \to \Gamma_\pi$ denotes the usual retraction.

**Lemma 5.2.** *Let $\pi$ be a good resolution of $(X,0)$ which factors through the the blowup of the maximal ideal and through the Nash transform of $(X,0)$. Then, for every $v$ in $\mathrm{NL}(X,0)$, we have*

$$\mathcal{I}_X(v) = \mathcal{I}_X\big(r(v)\big) + d\big(v, r_\pi(v)\big).$$

Since $\mathcal{I}_X$ is linear on the edges of $\Gamma_\pi$, this actually proves that $\mathcal{I}_X$ is completely determined by its restriction to the vertices of $\Gamma_\pi$.

*Proof.* Let $\pi'$ be the minimal good resolution of $(X,0)$ that factors through $\pi$ and such that $v$ is contained in $\Gamma_{\pi'}$. Observe that, if $\pi''$ is a good resolution of $(X,0)$ that is sandwiched in between $\pi'$ and $\pi$, so that $\Gamma_\pi \subset \Gamma_{\pi''} \subset \Gamma_{\pi'}$, then we have $d(v, r_\pi(v)) = d(v, r_{\pi''}(v)) + d(r_{\pi''}(v), r_\pi(v))$. This equality, together with the fact that $\pi'$ can be obtained from $\pi$ as a sequence of point blowups, allows us to reduce ourselves without loss of generality to the case where $\pi'$ is the composition of $\pi$ with the blowup of $X_\pi$ at a smooth point $p$ of the exceptional divisor $\pi^{-1}(0)$. Let $v_0$ denote the vertex of $\Gamma_\pi$ corresponding to the divisor $E_{v_0}$ containing $p$ and let $v_1$ denote the vertex of $\Gamma_{\pi'}$ corresponding to the exceptional divisor of the blowup at $p$, so that $v$ is a point of the edge $e = [v_0, v_1]$ of $\Gamma_{\pi'}$. Since $\mathcal{I}_X$ is linear on $e$, to prove the theorem it is sufficient to prove that $\mathcal{I}_X(v_1) = \mathcal{I}_X(v_0) + d(v_0, v_1)$. This follows immediately from the definition of $d$ and from Lemma 3.8.(i). □

**Remark 5.3.** Part of the proof of Corollary 5.1 can also be replaced by a purely combinatorial argument. Indeed, one can prove that if $F$ is a piecewise linear map on a metric graph $\Gamma$, then the Laplacian $\Delta_\Gamma(F)$ of $F$ determines $F$ uniquely up to an additive constant as follows. Assume that $F$ and $F'$ are two piecewise linear functions on $\Gamma$ with the same Laplacian, so that $G = F - F'$ is a piecewise linear function such that $\Delta_\Gamma(G) = 0$. In order to show that $G$ is constant one can reason by contradiction as follows. Assume that there exists an oriented segment $[v, v']$ in $\Gamma$ along which $G$ is strictly increasing. Then, since $\Delta_\Gamma(G)(v') = 0$, there exists a different segment $[v', v'']$ along which $G$ is strictly increasing as well. By iterating the same construction we obtain an infinite chain of segments along which $G$ grows.

As $\Gamma$ is finite, this yields a closed path along which $G$ strictly increases, which is absurd.

**Remark 5.4.** In earlier papers on the subject the inner rates where always computed by considering a generic projection $\ell\colon (X,0) \to (\mathbb{C}^2,0)$ and lifting the inner rates of the components of the exceptional divisor of a suitable resolution of the discriminant curve of $\ell$. For this reason, only simple examples have been computed, since it is generally very hard to decide whether a projection is generic, and moreover computing discriminant curves is not simple. Outside of the simplest examples, the calculations were usually done via computational tools such as Maple (see for example [BNP14, Example 15.2] or [SF18, Appendix]). However, it is generally much simpler to compute the data appearing in Corollary 5.1, and particularly so for hypersurfaces in $(\mathbb{C}^3,0)$. This means that our result often allows for a much simpler computation of the inner rates, as is the case in the next example where we compute the inner rates for Example 15.2 of [BNP14].

**Example 5.5.** Consider the hypersurface singularity $(X,0) \subset (\mathbb{C}^3,0)$ defined by the equation $(zx^2+y^3)(x^3+zy^2)+z^7=0$. A good resolution $\pi\colon X_\pi \to X$ of $(X,0)$ can be computed explicitly, we refer to [BNP14, Example 15.2] for the details. The exceptional divisor of $\pi$ is a configuration of copies $\mathbb{P}^1$ whose dual graph $\Gamma_\pi$ is drawn in Figure 5 represents the dual graph $\Gamma_\pi$. The vertices of $\Gamma_\pi$ are decorated with their serf-intersection, with solid arrows representing the components of the polar curve $\Pi$ of a generic projection of $(X,0)$, and with dashed arrows representing those of a generic hyperplane section.

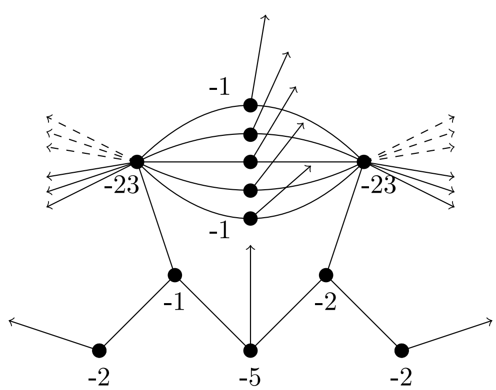



FIGURE 5.

Using the notations introduced in the proof of Corollary 5.1, we deduce from the data contained in Figure 5 the vectors $\underline{m} = -Ml$ and $\underline{b}$. We thus obtain the graph on the left of Figure 6, whose vertices are decorated by the pairs $(m_v, b_v)$. Finally, by computing $\underline{a} = {}^t(q_{v_1}m_{v_1}, \dots, q_{v_n}m_{v_n}) = M^{-1}\underline{b}$ and dividing each entry by the corresponding multiplicity $m_{v_i}$, we deduce the inner rates $q_{v_i}$, which are inscribed on the graph on the right of Figure 6.

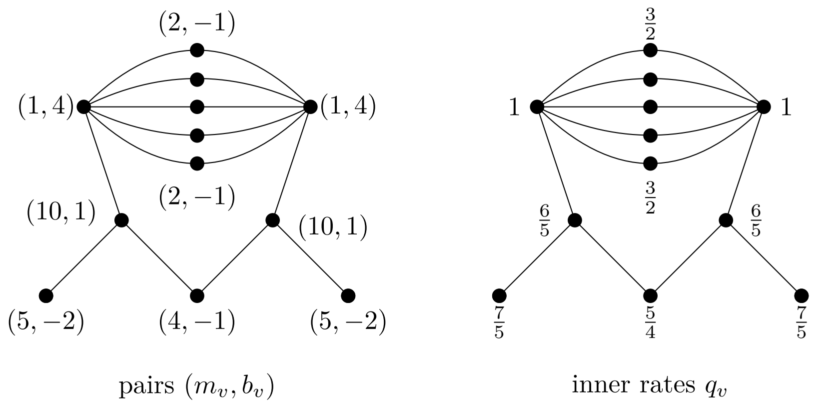



FIGURE 6.

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